Solutions for $\mathbb{E}$xercise sheet 2 : Construction and first properties of the Brownian motion.

Solution 1 — *Transformations.*
We first consider the finite-dimensional marginals of the new process $(X_t)_t$ in these three cases. Remark at first that they still form centered Gaussian vectors, since they are each obtained by a very simple linear transform of some f.d.m. of B. Now we only need to compute covariances.

(1) $\operatorname{Cov}(X_s, X_t) = \operatorname{Cov}(\lambda^{-1/2} B_{\lambda s}, \lambda^{-1/2} B_{\lambda t}) = \lambda^{-1} \operatorname{Cov}(B_{\lambda s}, B_{\lambda t}) = \lambda^{-1}(\lambda s \wedge \lambda t) = s \wedge t$.
(2) For $0 \leq s, t \leq 1$, $\operatorname{Cov}(X_s, X_t) = \operatorname{Cov}(B_1 - B_{1-s}, B_1 - B_{1-t}) = \operatorname{Cov}(B_1, B_1) - \operatorname{Cov}(B_1, B_{1-t}) - \operatorname{Cov}(B_{1-s}, B_1) + \operatorname{Cov}(B_{1-s}, B_{1-t}) = 1 - (1-t) - (1-s) + (1-s) \wedge (1-t) = 1 + (t-1) \wedge (s-1) = t \wedge s$.
(3) If $0 < s, t$, $\operatorname{Cov}(X_s, X_t) = \operatorname{Cov}(sB_{1/s}, tB_{1/t}) = st(s^{-1} \wedge t^{-1}) = t \wedge s$. If either $t = 0$ or $s = 0$, then we get $0 = s \wedge t$ for the covariance too.

Now the first two process are continuous on their whole domain, but for the third we only have continuity in $(0, \infty)$ and need to check continuity at 0 manually. We use a somewhat magic trick: We then consider the countable sequence $(X_t, t \geq 0, t \in \mathbb{Q}_+)$, $(B_t, t \geq 0, t \in \mathbb{Q}_+)$ which is a random variable in $(\mathbb{R}^{\mathbb{Q}_+}, \mathcal{B}(\mathbb{R})^{\otimes \mathbb{Q}_+})$. Because the finite dimensional marginals of B and X are the same, then it means that for a cylindrical set U,

$$(\spadesuit) \qquad \mathbb{P}(X_{|\mathbb{Q}_+} \in U) = \mathbb{P}(B_{|\mathbb{Q}_+} \in U).$$

Since cylindrical sets generate the product σ-algebra $\mathcal{B}(\mathbb{R})^{\otimes \mathbb{Q}_+}$, it means that $(\spadesuit)$ holds for all $U \in \mathcal{B}(\mathbb{R})^{\otimes \mathbb{Q}_+}$. Now consider the set $U = \left\{ A \in \mathbb{R}^{\mathbb{Q}_+} : A_t \xrightarrow[t \to 0^+, t \in \mathbb{Q}_+]{} 0 \right\}$. It belongs to $\mathcal{B}(\mathbb{R})^{\otimes \mathbb{Q}_+}$ because it can be written

$$\bigcap_{n \geq 1} \bigcup_{m \in \mathbb{N}} \bigcap_{q \in \mathbb{Q}_+ : q \leq 1/m} \{A : |A_q| < 1/n\}.$$

So $\mathbb{P}(X_{|\mathbb{Q}_+} \in U) = \mathbb{P}(B_{|\mathbb{Q}_+} \in U) = 1$ because B is continuous. Hence we have with probability one that:

(1) $t \mapsto X_t$ is continuous on $(0, \infty)$,
(2) $X_t \xrightarrow[t \to 0^+, t \in \mathbb{Q}]{} X_0$

So these with probability one we have the conjunction of these events, which implies continuity on the whole of $[0, \infty)$. Now if we change the X to the constant zero function whenever this event is not realized, this makes X continuous for all ω without changing the f.d.m's. So X is a Brownian motion.

Solution 2 — *Constructing a Brownian motion indexed by* $\mathbb{R}+$.
We can check continuity for all ω manually. Now a f.d.m. $B_{t_1}, \dots, B_{t_k}$ is a very simple linear transform of (some f.d.m. of $B^{(1)}$, some f.d.m. of $B^{(2)}$,..., some f.d.m. of $B^{(\lfloor t_k \rfloor)}$). Because of the independence assumption, this is a big Gaussian vector. Now we compute covariances. Let $s \le t$.

$$\begin{aligned}
\mathrm{Cov}(B_s, B_t) &= \mathrm{Cov}\left(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor} + \sum_{i=0}^{\lfloor s \rfloor -1} B^{(i)}_1, B^{(\lfloor \rfloor)}_{t-\lfloor t \rfloor} + \sum_{i=0}^{\lfloor t \rfloor -1} B^{(i)}_1 \right) \\
&= \sum_{i=0}^{\lfloor s \rfloor -1} \mathrm{Var}(B^{(i)}_1) + \mathrm{Cov}(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor}, B^{(\lfloor t \rfloor)}_{t-\lfloor t \rfloor}) \text{ if } \lfloor t \rfloor = \lfloor s \rfloor \\
&= \sum_{i=0}^{\lfloor s \rfloor -1} \mathrm{Var}(B^{(i)}_1) + \mathrm{Cov}(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor}, B^{(\lfloor s \rfloor)}_1) \text{ if } \lfloor t \rfloor > \lfloor s \rfloor \\
&= s \text{ anyway.}
\end{aligned}$$

This completes the proof.

Solution 3 — *A nowhere continuous version of the Brownian motion.*
Let $(X_t)_t$ be a Brownian motion and $(U_i)_i$ be an independent sequence of independent exponential random variables with parameter 1.
Let us show the following property: with probability one, $(U_i)_i$ is dense in $[0, \infty)$. Let $a < b \in \mathbb{Q}$. $\mathbb{P}(U_1 \notin [a,b], \dots, U_n \notin [a,b]) = \mathbb{P}(U_1 \notin [a,b])^n \to 0$ as $n \to \infty$. So $\mathbb{P}(U_i \notin [a,b] \forall i) = 0$. We have shown $\forall a < b \in \mathbb{Q}^2$, almost surely, $[a,b]$ intersects $(U_i)_i$. Because $\mathbb{Q}^2$ is countable, we can invert $\forall$ and almost surely, and we get that almost surely, $(U_i)_i$ is dense.
Now we define $B_t = X_t + \mathbb{1}_{t \notin \{U_i, i \in \mathbb{N}\}}$. By the previous property, this process is almost surely nowhere continuous, and we can check that the f.d.m's of B and X are equal almost surely (so have the same distribution) because for fixed $t_1, \dots, t_k$, the probability that $\{t_1, \dots, t_k\}$ intersects $\{U_i, i \in \mathbb{N}\}$ is 0 (once again by countable union).
Now we modify B on the negligible set where it is still continuous despite all our efforts, by setting $B = \mathbb{1}_{\mathbb{Q}}$, finishing the exercise.

Solution 4 — *Brownian motion is nowhere monotonous.*
Let us fix $a < b \in \mathbb{Q}$ and $a < t_0 < \ldots < t_k < b$. If the Brownian motion is increasing, it implies that $B_{t_i} - B_{t_{i_1}} \ge 0$ for every $1 \le i \le k$. So $\{B \text{ increasing on } [a,b]\} \subset \bigcap_{1 \le i \le k} \{B_{t_i} - B_{t_{i_1}} \ge 0\}$. This last event has probability 2^{-k} by independence. So $\{B \text{ increasing on } [a,b]\}$ can be included in an event of probability 0. So by countable union

$$\{B \text{ increasing on some interval }\} \subset \cup_{a<b \in \mathbb{Q}} \{B \text{ increasing on } [a,b]\}$$

can be included in an event of probability 0. So the complement property "B is increasing on no nontrivial interval" is almost sure. Same for "decreasing" by symmetry.

Remark: We did not need to show that the property "B is monotonous on no nontrivial interval" is indeed an event (i.e. is a measurable set), because the property of being almost sure or negligible can be defined for non-measurable subsets of Ω. But we can check that it is an event because of the assumption that paths are always continuous.

Solution 5 — *L^2 theory and construction of the Brownian motion.* (1) Immediate.

(2) Then setting $B_t = \langle \xi, I_t \rangle$ would yield a Gaussian process with the right covariance kernel. It can be checked by computing the characteristic function $(B_{t_1}, \dots, B_{t_k})$.

(3) Same computation: $\mathbb{E}[\exp(it_1 Z_{i_1} + \cdots + it_p Z_{i_p})] = \mathbb{E}[\exp(i\langle t_1 e_1 + \cdots + t_p e_p, \xi \rangle] = \prod_{i=1}^{p} e^{-it_p^2/2}$. Hence the distribution is that of i.i.d. standard Gaussians.

$$(\dagger) \qquad B_t = \langle \xi, I_t \rangle = \sum_{i=0}^{\infty} \langle \xi, e_i \rangle \langle I_t, e_i \rangle = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s) ds$$

(4) $\|\xi\|^2 = \sum_{i=0}^{\infty} Z_i^2$ which is a.s. not convergent because it does not go to 0 (Borel-Cantelli says that there exists a subsequence of i such that $Z_i > 0$ with probability 1).

(5) Indeed the primitives of the Haar wavelets are exactly the Schauder triangular functions that appear in Lévy's construction.

(6) We get $B_t = Z_0 t + \frac{\sqrt{2}}{\pi} \sum_{i=1}^{\infty} Z_m \frac{\sin(\pi m t)}{m}$.

(7) $\star$ ...