$\mathbb{E}$xercise sheet 3 : Regularity properties & miscellanea (v3)

Exercise 1 — *Boring but important measure theoretic stuff.*
For I a nonempty set, let $\mathcal{B}(\mathbb{R})^I$ be the product σ-algebra on $\mathbb{R}^I$.

(1) Recall the definition of $\mathcal{B}(\mathbb{R})^{\otimes I}$ and describe a π-system that generates it. Deduce that a probability measure on $(\mathbb{R}^I, \mathcal{B}(\mathbb{R})^{\otimes I})$ is characterized by the finite-dimensional marginals.
(2) Show that $(X_t)_{t\in I}$ is a random variable in $(\mathbb{R}^I, \mathcal{B}(\mathbb{R})^{\otimes I})$ if and only if X_t is a random variable in $\mathbb{R}$ for every t.
(3) Show that when I is countable, then $\mathcal{B}(\mathbb{R})^{\otimes I}$ is the Borel σ-algebra associated with the product topology on $\mathbb{R}^I$.
(4) Show that when I is not countable, then $\mathcal{B}(\mathbb{R})^{\otimes I} \subsetneq \mathcal{B}(\mathbb{R}^I)$.
(5) Show that the restriction of $\mathcal{B}(\mathbb{R})^{\otimes [0,1]}$ to $\mathcal{C}([0,1])$ is the Borel σ-algebra associated with the uniform convergence topology. Same with $\mathcal{C}(\mathbb{R}_+)$ and uniform convergence on every compact set.

Exercise 2 — *Indistinguishability and modifications.*
We recall that a stochastic process on some probaility space is a family of random variables X_t for t in some interval I. Two processes X, Y on I are said to be indistinguishable of each other if $\mathbb{P}(\forall t \in I,\ X_t = Y_t) = 1$. They are a modification of each other if $\forall t \in I$, $\mathbb{P}(X_t = Y_t) = 1$

(1) What is the relationship between these two notions ?
(2) Show that two continuous stochastic processes that are a modification of each other are actually indistinguishable.
(3) Show that "X is indistinguishable from a Brownian motion" is equivalent to "X is a $\mathcal{C}(\mathbb{R}_+)$-valued random variable (up to a.s. equality) which is distributed like a Brownian motion".

Exercise 3 — *Local regularity and long-term behavior.*
We recall from the last exercise session that $X(t) = tB(1/t)$, continued at 0 with $X(0) = 0$, is indistinguishable from a Brownian motion.

(1) Deduce that almost surely, $|B(t)| = o(t)$ as $t \to \infty$.
(2) Use bounds from the course to show that almost surely $\limsup_{t\to\infty} \frac{|B_t|}{t^{1/2-\epsilon}} = \infty$ for every $\epsilon > 0$, and that $B_t = O(\sqrt{t\log(t)})$ as $t \to \infty$, with a nonrandom $O(\cdot)$.
(3) We want to show that B is almost surely not 1/2-Hölder at 0. Fix $c > 0$.
   (a) Use Fatou's lemma to show that $\mathbb{P}(\limsup_n B_{2^{-n}}/\sqrt{2^{-n}} < c)$ is < 1.

(b) Show that $\{\limsup_n B_{2^{-n}}/\sqrt{2^{-n}} < c\}$ is a tail event for the sequence of independent random variables $(N_{n,1})_{n\geq 0}$ that appear in Lévy's construction.
(c) Conclude. What does it say for the long-term behavior ?

Exercise 4 — *Quadratic and absolute variation.*
For fixed t, a partition $\underline{t}$ is a finite sequence $0 = t_0 \leq t_1 \leq \ldots \leq t_{\#\underline{t}} = t$ and its mesh-size is $|\underline{t}| = \max_{1\leq i\leq \#\underline{t}} |t_i - t_{i-1}|$. The absolute (resp. quadratic) variation of B between 0 and t is

$$\lim_{\epsilon\to 0} \sup_{|\underline{t}|\leq\epsilon} \sum_{i=1}^{\#\underline{t}} |B_{t_i} - B_{t_{i-1}}| \qquad \left(\text{resp.} \quad \lim_{\epsilon\to 0} \sup_{|\underline{t}|\leq\epsilon} \sum_{i=1}^{\#\underline{t}} (B_{t_i} - B_{t_{i-1}})^2.\right)$$

(1) If $\underline{t}^{(k)}$ is a sequence of partitions with $|\underline{t}^{(k)}| \to 0$, then show that $\lim_{k\to\infty} \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2$ exists in the L^2 sense. What is it ?
(2) If $(\underline{t}^{(k)})_k$ is such that $\sum_{k=1}^{\infty} \sum_{j=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2 < \infty$, then the convergence is almost sure.
(3) Deduce that the Brownian motion almost surely does not have finite absolute variation (i.e. is not a bounded variation function).

Exercise 5 — *The precise constant (Lévy, 1937).*
We want to show that with probability one,

$$\limsup_{h\downarrow 0} \frac{m_B(h,[0,1])}{\sqrt{2h\log(1/h)}} = 1.$$

(1) Show that if X is standard Gaussian and $x > 0$, then

$$\frac{1}{\sqrt{2\pi}(x+1/x)} e^{-x^2/2} \leq \mathbb{P}(X \geq x) \leq \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}.$$

(2) For $c < \sqrt{2}$, show that almost surely for all $\epsilon > 0$ there exists $s,t \in [0,1]$ with $|t-s| \leq \epsilon$ and $|B(t) - B(s)| \geq c\sqrt{|t-s|\log(1/|t-s|)}$. (Hint: divide $[0,1]$ in intervals of length 2^{-n}).
(3) Fix $m \geq 1$ and define the following families of intervals:

$$\Lambda_n(m) = \left\{[(k/m-1)2^{-n/m}, (k/m)2^{-n/m}], \quad m \leq k \leq m2^{n/m}\right\}, \quad n \geq 1.$$

For $c > \sqrt{2}$, show that almost surely, for n large enough and any interval $[s,t]$ in the family $\Lambda_n(m)$, $|B(t) - B(s)| \leq c\sqrt{|t-s|\log(1/|t-s|)}$.
(4) Fix $\epsilon > 0$, show that there exists $m \geq 1$ such that any interval $[s,t] \subset [0,1]$ can be approximated with an interval $[s',t'] \in \Lambda(m) = \cup_{n\geq 1}\Lambda_n(m)$, with $|t-t'|, |s-s'| \leq \epsilon|t-s|$, and $|t'-s'| \leq |t-s|$.
(5) Deduce that almost surely, for h small enough, $m_B(h,[0,1]) \leq C\sqrt{h\log(1/h)}$, for a constant C that can be brought arbitrarily close to $\sqrt{2}$. Conclude.