$\mathbb{E}$xercise sheet 2 : Properties and construction of the Brownian Motion.

Exercise 1 — *Time inversion.*
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Set $X_t = tB_{1/t}$ for $t > 0$ and $X_0 = 0$.

(1) Show that X has the finite-dimensional marginals of a Brownian motion.
(2) What can you say about the set $U = \{A \in \mathbb{R}^{\mathbb{Q}_+}, \lim_{t\to 0, t\in\mathbb{Q}} A_t = 0\} \subset \mathbb{R}^{\mathbb{Q}_+}$?
(3) Deduce that $(X_t)_t$ is continuous almost surely, hence may be modified on a negligible event to form a Brownian motion.

Exercise 2 — *Constructing a Brownian motion indexed by* $\mathbb{R}+$.
Let $(B^{(n)})$ be a sequence of independent Brownian motions defined on $[0, 1]$. Define the following function $B : \mathbb{R}+ \to \mathbb{R}$:

$$B : t \mapsto B^{(\lfloor t\rfloor)}_{t-\lfloor t\rfloor} + \sum_{i=0}^{\lfloor t\rfloor -1} B^{(i)}_1.$$

Show that B is a Brownian motion.

Exercise 3 — *L^2 theory and construction of the Brownian motion.*
Let $H = L^2([0, 1])$ with the usual inner product. For $t \geq 0$ let $I_t = \mathbb{1}_{[0,t]} \in H$. We also set $(e_i)_{i\in\mathbb{N}}$ to be an orthonormal basis of H.

(1) Check that $\langle I_s, I_t\rangle = s \wedge t$.
(2) Suppose we could build a standard Gaussian random variable in H, that is $\xi \in H$ such that for every $x \in H$, $\langle x, \xi\rangle \sim \mathcal{N}(0, \|x\|)$. How could a Gaussian process $(B_t)_{t\in[0,1]}$ such that $\mathrm{Cov}(B_s, B_t) = s \wedge t$ be built from it ?
(3) Let $Z_i = \langle \xi, e_i\rangle$, so that $\xi = \sum_{i\in\mathbb{N}} Z_i e_i$. Show that the (Z_i) are independent standard Gaussians (*Hint :* compute the characteristic function of $(Z_{i_1}, \dots Z_{i_p})$ for $p \geq 1$ and $(i_1, \dots i_p) \in \mathbb{N}^p$). Deduce that we could then write the following equality in L^2:

$$B_t = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s)ds \qquad (\dagger)$$

(4) Show that ξ can not exist[1] (hint: compute its norm with the help of the basis e)

[1] It is possible to build ξ in the space $\mathcal{S}'$ of tempered distributions. It is then called a *white noise*, that is a random element of $\mathcal{S}'$ such that for every $\phi \in \mathcal{S} \subset L^2$, $\langle \phi, \xi\rangle \sim \mathcal{N}(0, \|\phi\|_2)$, see for instance [T.Hida, *Brownian Motion*, chapter 3, Springer 1980]

(5) Nevertheless, show that in the case of the Haar wavelet basis ef L^2: $h_0 = 1$ and for $n \geq 0$ and $0 \leq k < 2^n$

$$h_{k,n} := 2^{n/2}\left(\mathbb{1}_{[2k/2^{n+1},(2k+1)/2^{n+1}]} - \mathbb{1}_{[(2k+1)/2^{n+1},(2k+2)/2^{n+1}]}\right),$$

the series in (†) coincides with the Lévy construction of Brownian motion (and hence converges almost surely in $\mathcal{C}([0,1])$ to a Brownian motion).

(6) What do we obtain in (†) with the Fourier basis $e_0 = 1$, and $e_m(t) = \sqrt{2}\cos(\pi m t)$?

(7) ⋆ Show also the almost sure convergence in $\mathcal{C}([0,1])$ of this series.

Exercise 4 — *Brownian bridges.*

For $x, y \in \mathbb{R}$, we define the Brownian bridge of length one between x and y as follows: let B be a standard Brownian motion and set $\beta_t^{x,y} = x + B_t - tB_1 + t(y - x)$ for $t \in [0,1]$.

(1) Show that if X is a Brownian motion started from x, then the conditional distribution of the process $X_{|[0,1]}$ given $X_1 \in dy$ is $\beta^{x,y}$.

(2) For $0 < a < 1$, what expression does the Markov property applied at a give for the joint distribution of $(X_{|[0,a]}, X_1)$? Deduce an expression for the conditional distribution of $X_{|[0,a]}$ given $X_1 \in dy$. Deduce a second expression from the previous question.

(3) Show that the distribution of $\beta^{x,y}_{|[0,a]}$ is absolutely continuous with regard to that of $X_{|[0,a]}$ where $a < 1$.
Hint: use the fact that the conditional distributions are a.e. uniquely defined, along with some continuity argument.