Solutions for $\mathbb{E}$xercise sheet 2 : Construction and first properties of the Brownian motion.

Solution 1 — *Transformations.* (1) We first consider the finite-dimensional marginals of the new process $(X_t)_t$ in this case. Remark at first that they still form centered Gaussian vectors, since they are each obtained by a very simple linear transform of some f.d.m. of B. Now we only need to compute covariances: If $0 < s, t$, $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(sB_{1/s}, tB_{1/t}) = st(s^{-1} \wedge t^{-1}) = t \wedge s$. If either $t = 0$ or $s = 0$, then we get $0 = s \wedge t$ for the covariance too.

(2) Now consider the set $U = \left\{ A \in \mathbb{R}^{\mathbb{Q}_+} : A_t \xrightarrow[t \to 0^+, t \in \mathbb{Q}_+]{} 0 \right\}$. It can be written

$$\bigcap_{n \geq 1} \bigcup_{m \in \mathbb{N}} \bigcap_{q \in \mathbb{Q}_+ : q \leq 1/m} \{A : |A_q| < 1/n\},$$

hence it belongs to the σ-algebra generated by finite-dimensional sets.

(3) By the $\pi - \lambda$ (monotone class) theorem, two measures that coincide on a π-system Π (a family of sets stable by finite intersection), coincide on the generated σ-algebra $\sigma(\Pi)$. As a result, since B and X have the same finite-dimensional marginals, then $\mathbb{P}(X_{|\mathbb{Q}_+} \in U) = \mathbb{P}(B_{|\mathbb{Q}_+} \in U) = 1$. Hence we have with probability one that:

(a) $t \mapsto X_t$ is continuous on $(0, \infty)$,

(b) $X_t \xrightarrow[t \to 0^+, t \in \mathbb{Q}]{} X_0$

wich together implies continuity on the whole of $[0, \infty)$. Now if we change the X to the constant zero function whenever X is not continuous, this makes X continuous for all ω without changing the f.d.m's. So X is a Brownian motion.

Solution 2 — *Constructing a Brownian motion indexed by* $\mathbb{R}+$.
We can check continuity for all ω manually. Now a f.d.m. $B_{t_1}, \dots, B_{t_k}$ is a very simple linear transform of (some f.d.m. of $B^{(1)}$, some f.d.m. of $B^{(2)}$,..., some f.d.m. of $B^{(\lfloor t_k \rfloor)}$). Because of the independence assumption, this is a big Gaussian vector. Now we compute covariances. Let $s \leq t$.

$$\begin{aligned}
\mathrm{Cov}(B_s, B_t) &= \mathrm{Cov}\left(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor} + \sum_{i=0}^{\lfloor s \rfloor -1} B^{(i)}_1, B^{(\lfloor \rfloor)}_{t-\lfloor t \rfloor} + \sum_{i=0}^{\lfloor t \rfloor -1} B^{(i)}_1 \right) \\
&= \sum_{i=0}^{\lfloor s \rfloor -1} \mathrm{Var}(B^{(i)}_1) + \mathrm{Cov}(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor}, B^{(\lfloor t \rfloor)}_{t-\lfloor t \rfloor}) \text{ if } \lfloor t \rfloor = \lfloor s \rfloor \\
&= \sum_{i=0}^{\lfloor s \rfloor -1} \mathrm{Var}(B^{(i)}_1) + \mathrm{Cov}(B^{(\lfloor s \rfloor)}_{s-\lfloor s \rfloor}, B^{(\lfloor s \rfloor)}_1) \text{ if } \lfloor t \rfloor > \lfloor s \rfloor \\
&= s \text{ anyway.}
\end{aligned}$$

This completes the proof.

Solution 3 — L^2 *theory and construction of the Brownian motion.* (1) Immediate.

(2) Then setting $B_t = \langle \xi, I_t \rangle$ would yield a Gaussian process with the right covariance kernel. It can be checked by computing the characteristic function $(B_{t_1}, \dots, B_{t_k})$.
(3) Same computation: $\mathbb{E}[\exp(it_1 Z_{i_1} + \cdots + it_p Z_{i_p})] = \mathbb{E}[\exp(i\langle t_1 e_1 + \cdots + t_p e_p, \xi \rangle] = \prod_{i=1}^p e^{-it_p^2/2}$. Hence the distribution is that of i.i.d. standard Gaussians.

$$(\dagger) \qquad B_t = \langle \xi, I_t \rangle = \sum_{i=0}^{\infty} \langle \xi, e_i \rangle \langle I_t, e_i \rangle = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s) ds$$

(4) $\|\xi\|^2 = \sum_{i=0}^{\infty} Z_i^2$ which is a.s. not convergent because it does not go to 0 (Borel-Cantelli says that there exists a subsequence of i such that $Z_i > 0$ with probability 1).
(5) Indeed the primitives of the Haar wavelets are exactly the Schauder triangular functions that appear in Lévy's construction.
(6) We get $B_t = Z_0 t + \frac{\sqrt{2}}{\pi} \sum_{i=1}^{\infty} Z_m \frac{\sin(\pi m t)}{m}$.
(7) $\star$ ...

Solution 4 — *Brownian bridges.*
Here we denote $p(t, x, y)$ the Brownian transition kernel density $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)}$ for $t > 0$.

(1) Set $X_t = x + B_t$ and $\beta x, y_t = x + B_t - tB_1 + t(y - x)$. Remark that $\beta^{x,y}_t = \beta^{x,0}_t + yt$. But $X_t = \beta^{x,0}_t + tX_1$. Since $\beta^{x,0}$ is independent from X_1, it comes that $\mathbb{E}[H(X_{|[0,1]})|X_1] = \mathbb{E}[H(\beta^{x,0}_t + ty)]_{y=X_1} = \mathbb{E}[H(\beta^{x,y})]_{y=X_1}$. Hence the claim that $\mathbb{P}(X_{|[0,1]} \in \cdot | X_1 \in dy) = \mathbb{P}(\beta^{x,y} \in \cdot)$.

(2) The end of the exercise is essentially treated (in a different way) in the homework assignment.