Solutions for $\mathbb{E}$xercise sheet 3: stopping times and Markov property

Solution 1 — *Stopping times.*

Solution 2 — *Measurability of the stopped process.*

Solution 3 — *Counter-example.*

Solution 4 — *Another counter-example.*

(1) Consider the filtration $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$. Then we can write $X_{t+s} = X_t + (B_{t+s} - B_t) \mathbb{1}[A \neq 0]$. But almost surely, $\mathbb{1}[A \neq 0] = \mathbb{1}[X_t \neq 0]$, which means that we can rewrite $X_{t+s} = X_t + (B_{t+s} - B_t) \mathbb{1}[X_t \neq 0]$. Since $X_t \in \mathcal{F}_t$ and $B_{t+s} - B_t \perp\!\!\!\perp \mathcal{F}_t$, we get the Markov property with

$$p_t(x, dy) = \delta_0(dy) \text{ if } x = 0 \text{ and } \mathbb{P}(x + B_t \in dy) \text{ otherwise.}$$

(2) If it did, then it would mean that the process sticks to 0 after its first hitting time of 0, which is indeed not the case.

Solution 5 — *Brownian motion on the circle.*

Solution 6 — *The set of zeros of B is perfect.*
Almost surely 0 is an accumulation point of Z (lecture). By countable union, and strong Markov, every first 0 after any rational is an accumulation point of Z (at its right). If Z had an isolated point, it would be a first 0 after a rational. Hence it couldn't be isolated in Z.