Solutions for $\mathbb{E}$xercise sheet 4 : Markov processes and regularity properties

Solution 1 — *The stationary Ornstein-Uhlenbeck process.*
Firstly, $\mathrm{Cov}(X_t, X_s) = e^{-|t-s|}$. So at each time t, X_t is a standard Gaussian. For the Markov property, start from the standard filtration $\mathcal{F}$ of B. The standard filtration of X is then $t \mapsto \mathcal{F}_{e^{2t}}$. Then we easily compute

$$\mathbb{P}(X_{t+s} \in dy \mid \mathcal{F}_{e^2t}) = \mathbb{P}(e^{-s}x + B_{1-e^{-2s}} \in dy) = \frac{1}{\sqrt{2\pi(1-e^{-2s})}} \exp\left(\frac{(y-e^{-s}x)^2}{2(1-e^{-2s})}\right) dy.$$

Solution 2 — *Cauchy process.*
This exercise has been repurposed in exercise session 5, where compute the distribution through its Fourier transform thanks to a martingale argument. We give here the direct computation of the density for completeness.

(1) We have $C_a = B^{(2)}_{T_a} = \Psi(B^{(2)}, T_a)$ where $\Psi(\phi, t) = \phi_t$ is a measurable (actually continuous) functional $\mathcal{C}(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}$. Then for some positive measurable H,

$$\begin{aligned}
E[H(C_a)] = \mathbb{E}[H(\Psi(B^{(2)}, T_a))] &= \int_{R_+} \mathbb{P}(T_a \in dt) \int_{\mathcal{C}(\mathbb{R}_+)} \mathbb{P}(B^{(2)} \in d\phi) H(\Psi(\phi, t)) \\
&= \int_{R_+} \mathbb{P}(T_a \in dt) \int_{\mathcal{C}(\mathbb{R}_+)} \mathbb{P}(B \in d\phi) H(\phi_t) = \int_{R_+} \mathbb{P}(T_a \in dt) \int_{\mathbb{R}} \mathbb{P}(B_t \in du) H(u) \\
&= \int_{\mathbb{R}} H(u) \left(\int_{R_+} \frac{\mathbb{P}(T_a \in dt)}{dt} \frac{\mathbb{P}(B_t \in du)}{du} dt \right) du.
\end{aligned}$$

(Fubini has been used several times). Hence the thing inside the parentesis is the density of C_a at u. Let's compute it

$$\begin{aligned}
\frac{\mathbb{P}(C_a \in du)}{du} &= \int_{R_+} \frac{\mathbb{P}(T_a \in dt)}{dt} \frac{\mathbb{P}(B_t \in du)}{du} dt \\
&= \int_{R_+} \frac{a}{\sqrt{2\pi} t^{3/2}} e^{-a^2/2t} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dt = \frac{a}{\pi(x^2 + a^2)}
\end{aligned}$$

Solution 3 — *One-sided and two-sided local minima.*
A (strict) local minimum of B is a time t for which there exists $\varepsilon > 0$ such that $B_s \leq B_t, s \in [t-\varepsilon, t+\varepsilon]$ ($B_s < B_t, s \in [t-\varepsilon, t+\varepsilon], s \neq t$). A right decrease point is a time t for which there exists $\varepsilon > 0$ such that $B_s < B_t, s \in [t, t+\varepsilon]$.

(1) We know that almost surely 0 is not an increase point nor a decrease point at its right. By Markov this is also the case for any given point t. But now

$$\mathbb{E}[\mathrm{Leb}\{t \geq 0, t \text{ special}\}] = \int_{\mathbb{R}} dt\, \mathbb{P}(t \text{ special}) = \int_{\mathbb{R}} 0 = 0$$

Hence this positive random variable is almost surely 0.

(2) Almost surely a given time t is not a point of increase at its right. By time reversal $s > t$ is almost surely not a point of incease at its left. Hence the global minimum of B on $[s,t]$ is reached in the interior (s,t), and forms a local minimum. Taking the countable union for $s,t \in \mathbb{Q}$ gives that almost surely a density of local minima exists.

(3) Let $a < b < c < d$. Let us consider $U = B_b - \min_{[a,b]} B$, $V = B_c - B_b$, $W = B_c - \min_{[c,d]} B$. By Markov at b, $U \perp\!\!\!\perp (V,W)$. By Markov at c, $W \perp\!\!\!\perp V$. Hence (U,V,W) forms an independent triple.

Now we have $U \sim |B_{b-a}|$, $V \sim B_{c-b}$, $W \sim |B_{d-c}|$. All those random variables admit a density. The event that the two minima coincide is the event $\{U+V = W\}$, which then has probaility 0.

(4) Assume the existence of a non-strict local minimum. Then there exists a x and ϵ such that $B_y \geq B_x$ for all $y \in [x-\epsilon, x+\epsilon]$. Moreover, for all $\delta > 0$, we may find $z \in [x-\delta, x+\delta]$ such that $B_z = B_x$. Take such a z for $\delta < \epsilon$. Then we may find $a,b,c,q \in \mathbb{Q}$, $x - \epsilon < a < x < b < c < z < d < x + \epsilon$. Then the minimum value of B on both $[a,b]$ and $[c,d]$ is B_x. By the previous question the probability of this event is 0.

The fact the the set of strict local minima of a function is countable is deterministic: each of them is the unique argmin on a rational interval.

(5) For every $a \in [B_1, \max_{[0,1]} B]$, the time $\sup\{t \in [0,1], B_t = a\}$ is a right decrease point. All of them are distinct for distinct a. Hence the uncountability.

Solution 4 — *Quadratic and absolute variation.* (1) We first try to guess what the L^2 limit could be. If we have L^2 convergence to say X, since L^2 convergence implies L^1 convergence, we get that $\mathbb{E}[X] = \lim_k \mathbb{E}[\sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2] = \lim_k t = t$ as we observe a telescoping series. So $\mathbb{E}[X] = t$ and if by any chance X was deterministic[1], we'd have $X = t$. Let's try to show convergence to t. We rewrite

$$A_k = \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 - t = \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 - (t_i^{(k)} - t_{i-1}^{(k)}).$$

[1]Reasonable because it seems that there could be 0-1 law w.r.t. Lévy construction

Then

$$\mathbb{E}[A_k^2] = \operatorname{Var}(A_k) = \sum_{i=1}^{\#\underline{t}^{(k)}} \operatorname{Var}((B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 - (t_i^{(k)} - t_{i-1}^{(k)}))$$

$$= \sum_{i=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2 \operatorname{Var}(Z^2 - 1) = (\mathrm{cst}) \sum_{i=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2. \quad \leq (\mathrm{cst})\, t|\underline{t}^{(k)}|$$

which goes to 0. We have moved freely between variance and second moment because everything is centered, used the independence of increments and the decomposition of variance over an independent sum, the fact that a standard Gaussian Z has a fourth moment, and Hölder$(1,\infty)$ at the end. We have $A_k \to 0$ in L^2 which is the claim.

(2) If $(\underline{t}^{(k)})_k$ is such that $\sum_{k=1}^{\infty} \sum_{j=1}^{\#|\underline{t}|} (t_i - t_{i-1})^2 < \infty$, then we get $\sum_{k=1\infty} \mathbb{E}[A_k^2] < \infty$. For $\epsilon \in \mathbb{Q}_+^*$ we have $\mathbb{P}(|A_k| > \epsilon) \leq \epsilon{-}2\,\mathbb{E}[A_k^2]$ which is summable. So by Borel-Cantelli, almost surely, for large n, $|A_k| < \epsilon$. We invert " $\forall \epsilon \in \mathbb{Q}_+^*$" and "almost surely" by countable union and we're done.

(3) If B has bounded variation, then it is not hard to show that the quadratic variation is 0 (once again Hölder$(1,\infty)$). But this is a.s. impossible because of the previous question.