ENS de Lyon — Mathematic department | Master 1 — Spring 2023
Stochastic processes | E. Jacob

FINAL EXAM ANSWER KEY.

Tuesday, April 25. 2pm-5pm.

You may answer in either english or french. Your lecture notes are authorized, but we remind below some useful results:

- *If B is standard one-dimensional Brownian motion and $\lambda \in \mathbb{R}$, then the process*
$$\left(e^{\lambda B_t - \frac{\lambda^2}{2}t}\right)_{t\geq 0}$$
is a martingale.
- *If B is a d-dimensional Brownian motion with $d \geq 3$ started from some x with $|x| > r > 0$, then*
$$\mathbb{P}_x(\exists t \geq 0, |B_t| = r) = \left(\frac{r}{|x|}\right)^{d-2}.$$
- *If B is standard one-dimensional Brownian motion starting from 0 and $T_x = \inf\{t \geq 0, B_t = x\}$ for some $x \in \mathbb{R}$, then T_x has the same law as x^2/N^2, where N is a standard centered gaussian random variable.*

Exercice 1 — *Hitting an affine line.*
Let (B_t) be a standard one-dimensional Brownian motion started from 0. For $a, b > 0$, we write
$$T := \inf\{t \geq 0, B_t = at + b\}.$$

(1) Compute the probability of T being finite.
Answer: We consider the exponential martingale
$$M_t := e^{2aB_t - 2a^2 t}.$$
We know that M_t/t tends to 0 a.s., and thus M_t tends a.s. to $M_\infty = 0$. The process $(M_{t\wedge T})_{t\geq 0}$ is a nonegative martingale, started from $M_0 = 1$ and bounded by e^{2ab}, using the inequality $B_t \leq at + b$ which is satisfied for $t \leq T$. Thus it is a uniformly integrable martingale, closed by its almost sure limit $M_T = M_T \mathbb{1}_{T<+\infty} = e^{2ab}\mathbb{1}_{T<+\infty}$. The stopping theorem for closed martingales gives $\mathbb{E}[M_T] = 1$ and thus
$$\mathbb{P}(T < +\infty) = e^{-2ab}.$$

(2) Deduce the value of $\mathbb{E}[\sup\{B_t - at, t \geq 0\}]$.
Answer: We deduce that $\sup\{B_t - at, t \geq 0\}$ is exponentially distributed with parameter $2a$ and thus has expectation $1/2a$.

Exercice 2 — *Hitting a high-dimensional curve.*
Let (B_t) be a Brownian motion started from 0 in $\mathbb{R}^d$ for some $d \geq 4$. Let $f : [0,1] \to \mathbb{R}^d \backslash \{0\}$ be a (deterministic) function assumed to be $\alpha-$Hölder for some $\alpha \in (0,1]$, namely

$$\exists C > 0, \quad \forall 0 \leq s, t \leq 1, \quad |f(t) - f(s)| \leq C|t-s|^\alpha.$$

(1) Under the condition $\alpha(d-2) > 1$, show that the Brownian motion a.s. never hits the image of f.

Hint: For $n \geq 1$ large, cover the image of f by n balls of radius at most $C(2n)^{-\alpha}$.

Answer: Write $r = \min\{|f(t)|, 0 \leq t \leq 1\}$, and consider n large enough so that $r_n := C(2n)^{-\alpha} < r$. For $1 \leq k \leq n$, write D_k for the closed ball centered at $f((2k-1)/2n)$ and of radius r_n, so that the image of f is included in $\cup_{1 \leq k \leq n} D_k$. We have

$$\begin{aligned} \mathbb{P}(T_{\Im f} < \infty) &\leq \sum_k \mathbb{P}(T_{D_k} < +\infty) \\ &\leq \sum_k \left(\frac{r_n}{r}\right)^{d-2} \\ &\leq n \left(\frac{r_n}{r}\right)^{d-2}, \end{aligned}$$

which tends to 0 as $n \to \infty$ under the hypothesis $\alpha(d-2) > 1$, whence the result.

(2) Deduce that in dimension $d \geq 5$, two independent Brownian motions B and $\tilde{B}$ with distinct starting points almost surely have nonintersecting images, namely:

$$\mathbb{P}(\exists s, t \geq 0, B_s = \tilde{B}_t) = 0.$$

Answer: By translation, we can suppose B starts from 0 and $\tilde{B}$ has a different starting condition. Then, almost surely, the function $\tilde{B} : \mathbb{R}_+ \to \mathbb{R}^d$ never hits 0 and is $\alpha-$Hölder on every time interval $[k, k+1]$ with $k \in \mathbb{N}$ and $\alpha = 2/5 < 1/2$. The condition $\alpha(d-2) > 1$ is satisfied, and we can then apply the previous question to deduce that the brownian motion B a.s. never hits the image of $B_{|[k,k+1]}$, whence the result.

Exercice 3 — *An extension of Liouville's theorem.*
In this exercice we work in dimension $d \geq 2$ and consider a point $\mathbf{z} = (r, 0, \ldots, 0) \in \mathbb{R}^d$ for some $r > 0$. We suppose that $(B_t)_{t \geq 0}$ is a d-dimensional Brownian motion started from $\mathbf{z}$ under the probability measure $\mathbb{P}_\mathbf{z}$. We will write x_i for the i-th coordinate of a point $\mathbf{x} \in \mathbb{R}^d$, and $|\mathbf{x}|$ for its euclidean norm, namely $\mathbf{x} = (x_1, \ldots, x_d)$ and $|\mathbf{x}| = (x_1^2 + \ldots x_d^2)^{1/2}$. For $R > r$, we consider the hyperplanes

$$\begin{aligned} H &:= \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, x_1 = 0\}, \\ H_R &:= \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, x_1 = R\}, \\ L_R &:= \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, x_2 = R\}. \end{aligned}$$

We also consider the sphere $C_{R\sqrt{d}} := \{\mathbf{x} \in \mathbb{R}^d, |\mathbf{x}| = R\sqrt{d}\}$. We write T_H for the hitting time H, and similarly T_{H_R}, aso.

(1) Compute the probability $\mathbb{P}_{\mathbf{z}}(T_{H_R} < T_H)$.

Answer: This is the probability that the first coordinate, started from r, reaches R before reaching 0, famously known to be equal to r/R.

(2) Show that we have

$$\mathbb{P}_{\mathbf{z}}(T_{L_R} < T_H) = \frac{2}{\pi} \arctan \frac{r}{R}.$$

Answer: Observe that T_H is the hitting-time of 0 for the first coordinate (started from r), and T_{L_R} is the hitting-time of R for the second coordinate (started from 0). The two coordinates are independent, and we thus can compute

$$\mathbb{P}_{\mathbf{z}}(T_{L_R} < T_H) = \mathbb{P}(\frac{R^2}{N^2} < \frac{r^2}{N'^2}),$$

where N and N' are two independent standard gaussian random variables. If we write in polar coordinates $(N, N') = (X\cos\theta, X\sin\theta)$ with $X \geq 0$ and $\theta \in (-\pi, \pi]$, then θ is uniform, as the law of (N, N') (which is also $\mathcal{N}(0, I_2)$) is invariant under the isometries of $\mathbb{R}^2$. We can thus further compute

$$\mathbb{P}_{\mathbf{z}}(T_{L_R} < T_H) = \mathbb{P}(|\tan\theta| \leq \frac{r}{R}) = \frac{2}{\pi} \arctan \frac{r}{R}.$$

(3) Deduce the upper bound

$$\mathbb{P}_{\mathbf{z}}(T_{C_{R\sqrt{d}}} < T_H) \leq \frac{r}{R} + \frac{4(d-1)}{\pi} \arctan \frac{r}{R}.$$

If the process hits $C_{R\sqrt{d}}$ before H, one of its coordinate has to hit the value R or $-R$ before the process hits H. The probability that the first coordinate hits R before time T_H is r/R by question (1), and the probability it hits $-R$ is before time T_H is of course 0. For any other coordinate, the probability that it hits R or $-R$ before time T_H is bounded by $\frac{4}{\pi} \arctan \frac{r}{R}$ by question (2). The result follows.

(4) If h is a harmonic function on $\mathbb{R}^d$ satisfying $\frac{|h(\mathbf{x})|}{|\mathbf{x}|} \to 0$ as $|\mathbf{x}| \to +\infty$, show that h is constant.

Answer: We proceed as in the proof of Liouville's theorem. Take h as in the question, and choose $y \neq z$ in $\mathbb{R}^d$, and x the center of the segment $[yz]$. We aim to show $h(y) = h(z)$. By spatial translation and an isometry, we can suppose $z = (r, 0, \ldots, 0)$ and $y = -z$. For $R > r$, we let D_R be the half-disk domain containing z and delimited by H and $C_{R\sqrt{d}}$. We then have

$$\begin{aligned} h(z) &= \mathbb{E}_z[h(B_{T_{\partial D_R}})] \\ &= \mathbb{E}_z[h(B_{T_H})\,\mathbb{1}_{T_H < T_{C_{R\sqrt{d}}}}] + \mathbb{E}_z[h(B_{T_{C_{R\sqrt{d}}}})\,\mathbb{1}_{T_{C_{R\sqrt{d}}} < T_H}] \\ &= \mathbb{E}_y[h(B_{T_H})\,\mathbb{1}_{T_H < T_{C_{R\sqrt{d}}}}] + \sup\{|h(x)|, |x| = R\sqrt{d}\}\,\mathbb{P}_z(T_{C_{R\sqrt{d}}} < T_H) \end{aligned}$$

where we used an obvious symmetry of the problem to replace z by y in the last equality. We have a similar expression for $h(y)$, and it remains to prove that the second term after the last equality tends to 0 as $R \to +\infty$. This follows from last question and the hypothesis $\sup\{|h(x)|, |x| = R\sqrt{d}\} = o(R)$.

Exercice 4 — *Azéma-Yor embedding.*
Let B be a one-dimensional Brownian motion. Given a real-valued random variable X with $\mathbb{E}[X] = 0$ and $\operatorname{Var} X < +\infty$, Skorokhod embedding problem stems at finding some stopping-time T with $\mathbb{E}[T] < +\infty$ such that B_T and X have the same law.

(1) We first suppose $\mathbb{P}_X = \frac{1}{4}\delta_{-3} + \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_2$.

(a) We define $S := \inf\{t \geq 0, B_t \notin (-3, 1)\}$ and $T := \inf\{t \geq S, B_t \notin (-1, 2)\}$. Show that T is a solution to Skorokhod embedding problem.
Answer: We have $\mathbb{P}(B_S = -3) = 1/4$ and $\mathbb{P}(B_S = 1) = 3/4$. Further, on the event $B_S = -3$, we have $T = S$ and $B_T = -3$. On the event $B_S = 1$, we have $B_T \in \{-1, 2\}$, with

$$\mathbb{P}(B_T = 2) = \mathbb{P}(B_S = 1)\,\mathbb{P}(B_T = 2|\ B_S = 1) = \frac{3}{4}\frac{2}{3} = \frac{1}{2}.$$

Finally the law of B_T is $\mathbb{P}_X$. Furthermore, $\mathbb{E}[T] < +\infty$, for example because $T \leq T_{\{-3,2\}}$. By Wald's second lemma, we then necessarily have $\mathbb{E}[T] = \mathbb{E}[B_T^2] = \mathbb{E}[X^2]$.

(b) Write explicitly the similar construction of the solution $\tilde{T}$ of Skorokhod embedding problem as provided by the approach seen in the lecture. Do we have $\mathbb{E}[\tilde{T}] = \mathbb{E}[T]$? Do you think T and $\tilde{T}$ have the same law?
Answer: In the construction given in the lecture, we first decide wether we will take a positive or a negative value. We then take $\tilde{S}$ the hitting time of $\{-2, 2\}$ and then

$$\tilde{T} = \inf\{t \geq \tilde{S}, B_T \in \{-3, -1, 2\}\}.$$

We see the two constructions differ, and there is no reason to believe that T and $\tilde{T}$ have the same law, however $\mathbb{E}[T] = \mathbb{E}[\tilde{T}] = \mathbb{E}[X^2]$.

(2) We suppose now that X (is still centered and) takes only finitely many values

$$x_1 < \ldots < x_n.$$

For $0 \leq k \leq n-1$, define $y_k := \mathbb{E}[X|\ X \geq x_{k+1}]$ and

$$Y_k := \begin{cases} y_k & \text{if } X \geq x_{k+1} \\ X & \text{if } X \leq x_k, \end{cases}$$

so in particular $Y_0 = y_0 = 0$, while $y_{n-1} = x_n$ and $Y_{n-1} = X$.

(a) Show that $(Y_k)_{0 \leq k \leq n-1}$ is a martingale.
Answer: Defining $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(\{X \leq x_i, i \leq k\})$, we see that $(\mathcal{F}_k)_{0 \leq k \leq n-1}$ is a filtration and $Y = \mathbb{E}[X|\mathcal{F}_k]$, whence Y is a martingale.

(b) We define recursively the stopping times $T_0 = 0$ and for $1 \leq k \leq n-1$,

$$T_k = \inf\{t \geq T_{k-1}, B_t \notin (x_k, y_k)\}.$$

For $1 \leq k \leq n-1$, show that the law of B_{T_k} given $B_{T_{k-1}} = y_{k-1}$ coincides with the law of Y_k given $Y_{k-1} = y_{k-1}$. Deduce that the random variables B_{T_k} and Y_k have the same law, and then T_{n-1} is a solution to Skorokhod embedding problem.

Answer: The law of B_{T_k} given $B_{T_{k-1}} = y_{k-1}$ has support $\{x_k, y_k\}$ and expectation y_{k-1}. The same holds for the law of Y_k given $Y_{k-1} = y_{k-1}$. Thus the two conditional laws coincide. Furthermore, on the event $B_{T_{k-1}} < y_{k-1}$, we have $B_{T_k} = B_{T_{k-1}}$, and again a similar statement holds for Y. We deduce that the processes $(Y_k)_{0\leq k\leq n-1}$ and $(B_{T_k})_{0\leq k\leq n-1}$ have the same law. In particular, $B_{T_{n-1}}$ has the same law as X, and $\mathbb{E}[T_{n-1}] < +\infty$, for example because $T_{n-1} \leq T_{\{x_1,x_n\}}$.

(c) Show that an equivalent definition of T_{n-1} is

$$T_{n-1} := \inf\{t \geq 0, B_t^\star \geq \psi(B_t)\},$$

where $\psi(x)$ is defined as $\psi(x) = \mathbb{E}[X|\ X \geq x]$ if $\mathbb{P}(X \geq x) > 0$, and $\psi(x) = 0$ otherwise.

Hint: To this end, you may observe that on the event $\{B_{T_{n-1}} = x_k\}$ for some $k \leq n-1$, we have $T_{n-1} = T_k$, and consider separately times $t \in [T_{i-1}, T_i)$ for $i \leq k$ and time T_k.

Answer: We first work on the event $B_{T_{n-1}} = x_k$ for $k \leq n-1$, as in the hint. We then have $T_k = T_{n-1}$, as well as $B_{T_i} = y_i$ for $i < k$ and $B_{T_k} = x_k$. It follows that for times t in $[T_{i-1}, T_i)$, we have $B_t^\star \in [y_{i-1}, y_i)$. We now treat the two cases separately: For $t \in [T_{i-1}, T_i)$ with $i < k$, we have $B_t > x_i$ and thus $\psi(B_t) \geq \psi(x_{i+1}) = y_i$, while $B_t^\star < y_i$, and thus $B_t^\star < \psi(B_t)$. At time T_k, we have $B_{T_k} = x_k$ and thus $\psi(B_{T_k}) = y_{k-1}$, while $B_{T_k}^\star \geq y_{k-1}$. Whence the result.

(3) In the general case, we still define $\psi(x)$ as $\psi(x) = \mathbb{E}[X|\ X \geq x]$ if $\mathbb{P}(X \geq x) > 0$, and $\psi(x) = 0$ otherwise. We admit that there is a sequence or centered random variables (X_n) taking only finitely many values such that X_n converges to X in distribution and τ_n converges almost surely to τ, where

$$\psi_n(x) = \begin{cases} \mathbb{E}[X_n|\ X_n \geq x] & \text{if } \mathbb{P}(X_n \geq x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$
$$\tau_n = \inf\{t \geq 0, B_t^\star \geq \psi_n(B_t)\},$$
$$\tau = \inf\{t \geq 0, B_t^\star \geq \psi(B_t)\}.$$

Show that τ is a solution to Skorokhod embedding problem.

Answer: By previous work, we have that B_{τ_n} has the same law as X_n and $\mathbb{E}[\tau_n] = \mathbb{E}[X_n^2]$. We admit that we also can request $\mathbb{E}[X_n^2] \to \mathbb{E}[X^2]$. As τ_n tends to τ a.s., we have that B_{τ_n} tends to B_τ a.s. and thus in law, thus B_τ has the same law as X.

Furthermore, by Fatou lemma,

$$\mathbb{E}[\tau] \le \liminf \mathbb{E}[\tau_n] = \liminf \mathbb{E}[X_n^2] = \mathbb{E}[X^2] < +\infty.$$

Exercice 5 — *Estimate of the tail of a random walk hitting time.*
In this exercice, we consider S_n a simple random walk on $\mathbb{Z}$, supposed to be started from x under the probability measure $\mathbb{P}_x$. For $a \in \mathbb{Z}$, we write T_a for the hitting time $T_a := \inf\{n \ge 0, S_n = a\}$. The main purpose is to obtain estimates on the probability of the tail event $\{T_{-1} > n\}$ with the help of Brownian motion.

(1) (a) For $x \in \mathbb{N}$, recall briefly why we have

$$\mathbb{P}_0(T_x < T_{-1}) = \frac{1}{1+x}.$$

Answer: This is again gambler's ruin problem, but in the context of the random walk.

(b) Show $(S_n^2 - n)$ is a martingale, and deduce $\mathbb{E}_0[T_x \wedge T_{-1}] = x$ and further, for $t \in \mathbb{N}^*$,

$$\mathbb{P}_0(t \le T_x \wedge T_{-1}) \le \frac{x}{t}.$$

(2) We suppose $x_n \sim a\sqrt{n}$ and $t_n \sim bn$, with $a, b \in \mathbb{R}_+^*$.

(a) Show that we have

$$\mathbb{P}_{x_n}(T_{-1} > t_n) = \mathbb{P}_0(T_{-x_n-1} > t_n) \to \mathbb{P}(\inf\{B_t, 0 \le t \le b\} \le -a),$$

where B is a standard 1-dimensional brownian motion started from 0.
Hint: You may first suppose $t_n = n$ and argue that $\mathbb{P}(\inf\{B_t, 0 \le t \le 1\} = -a) = 0$.
Answer: We apply Donsker theorem, first with $x_n = a\sqrt{n}$ and $t_n = bn$...

(b) Show that we also have

$$\mathbb{P}(\inf\{B_t, 0 \le t \le b\} \le -a) = \mathbb{P}(|N| \le \frac{a}{\sqrt{b}}) \sim \sqrt{\frac{2}{\pi}}\,\frac{a}{\sqrt{b}},$$

where N is standard gaussian and the equivalent is as $a/\sqrt{b}$ tends to 0.
Answer: For the first equality, we use that

$$|\inf\{B_t, 0 \le t \le b\}| \overset{law}{=} |B_b| \overset{law}{=} \sqrt{b}|N|.$$

The equivalent follows from the fact that the density of the normal distribution at 0 is $1/\sqrt{2\pi}$.

(3) We now consider the random walk started from 0. For given $x_n \ge 0$ and $t_n \ge 0$, justify the inclusions of events

$$\{T_{-1} > n\} \supset \{T_{x_n} < T_{-1}\} \cap \{T_{-1}(S^{(T_{x_n})}) > n\},$$

$$\{T_{-1} > n\} \subset \big(\{T_{x_n} < T_{-1}\} \cap \{T_{-1}(S^{(T_{x_n})}) > n - t_n\}\big) \cup \{t_n \le T_{x_n} \wedge T_{-1}\},$$

where $S^{(t)}$ is the usual notation for the process $(S_{n+t})_{n\geq 0}$, and $T_x(S^{(t)})$ is its hitting time of x.

Answer: For the first inclusion, we simply observe that the event in the RHS implies that the process stays nonnegative at least until time $T_{x_n}+n>n$.
The second inclusion misses the hypothesis $t_n \leq n$. On the event $T_{-1}<n$, either the process stays within $[0, x_n-1]$ up until time t_n, or the process hits x_n before time t_n, and then after time T_{x_n} it has to stay nonnegative up until time $n-T_{x_n} \geq n-t_n$, whence the second inclusion.

(4) Choosing, for $\varepsilon>0$, sequences (x_n) and (t_n) that satisfy $x_n \sim \varepsilon\sqrt{n}$ and $t_n \sim \sqrt{\varepsilon}n$, deduce that we have

$$\mathbb{P}_0(T_{-1}>n) \sim \sqrt{\frac{2}{\pi n}}$$

Answer: We choose $\varepsilon>0$ and use the first inclusion, in which the events in the RHS are independent, to deduce

$$\mathbb{P}(T_{-1}>n) \geq \frac{1}{1+x_n}\mathbb{P}_{x_n}(T_{-1}>n).$$

We deduce

$$\liminf \sqrt{n}\,\mathbb{P}(T_{-1}>n) \geq \frac{1}{\varepsilon}\mathbb{P}(|N| \leq \varepsilon).$$

Taking $\varepsilon \to 0$, we deduce the lower bound $\sqrt{2/\pi}$.
Similarly, the second inclusion provides the upper bound

$$\limsup \sqrt{n}\,\mathbb{P}(T_{-1}>n) \leq \frac{1}{\varepsilon}\mathbb{P}\left(|N| \leq \frac{\varepsilon}{\sqrt{1-\sqrt{\varepsilon}}}\right)+\sqrt{\varepsilon}.$$

Taking $\varepsilon \to 0$ provides the upper bound $\sqrt{2/\pi}$ and allows to conclude.

(5) Provide similar asymptotics for $\mathbb{P}_0(T_{-k}>n)$ for given $k>0$.

In this question and the next, you may skip details and just explain briefly how to adapt the proof to that case.

(6) Treat similarly the case of any random walk whose jump distribution is centered and supported on $\{-1, 0, 1, \ldots, l\}$ for some finite $l \geq 1$.