PARTIAL EXAM.

Tuesday, February 28. 10.15am-12.15pm.

Usual notations: CTMC for continuous-time Markov chain or equivalently PJMP for Pure Jump Markov process.
The two exercices are independent.

Exercice 1 — *Explosion for the accelerated biased random walk, and duality.*
We consider $(Y_n)_{n\geq 0}$ the biased random walk with positive drift, which is the discrete-time Markov chain with jump distribution $p\delta_1 + q\delta_{-1}$, where $q = 1-p$ is assumed to be strictly smaller than p, or equivalently $p > 1/2$. We recall that this random walk is transient, and $\mathbb{P}(Y_n \to +\infty) = 1$. As usually, we suppose that under the probability measure $\mathbb{P}_0$, this biased random walk is started from 0.

(1) Recall briefly why we have

$$\mathbb{P}_0(T_{-1} < +\infty) = \frac{q}{p}, \quad \mathbb{P}_0(H_0 < +\infty) = 2q,$$

where T_{-1} is the first hitting time of -1, and H_0 the first return time to 0, defined as $\inf\{n \geq 1, Y_n = 0\}$.

Advice: do not hesitate to admit the first statement or even the two if needed, it will not prevent you from answering the following questions.

(2) Show the Green function of this process is equal to

$$G_Y(0,n) = \begin{cases} \frac{1}{1-2q} & \text{if } n \geq 0, \\ \left(\frac{q}{p}\right)^{|n|} \frac{1}{1-2q} & \text{if } n < 0. \end{cases}$$

We consider now $(q_n)_{n\in\mathbb{Z}} \in [1,+\infty)^{\mathbb{Z}}$ an arbitrary sequence of real numbers larger than or equal to 1 and indexed by $\mathbb{Z}$, and $(X_t)_{t\geq 0}$ the accelerated biased random walk. It is the pure jump Markov process with associated jump process the biased random walk Y we introduced before, and whose waiting time at state n is distributed as an exponential random variable with parameter q_n.

(3) Show that the process (X_t) does not explode if $\sum_{n\geq 0} \frac{1}{q_n} = +\infty$.

(4) Letting ζ be the explosion time and $G(\cdot,\cdot)$ the Green function of the CTMC X, show we have

$$\mathbb{E}_0[\zeta] = \sum_{n\in\mathbb{Z}} G(0,n).$$

Compute this sum, and deduce that the process explodes if $\sum_{n\geq 0} \frac{1}{q_n} < +\infty$.

(5) Show the measures ν and $\tilde{\nu}$ on $\mathbb{Z}$ defined by

$$\nu_n = 1, \qquad \tilde{\nu}_n = \left(\frac{p}{q}\right)^n,$$

are invariant for Y, and deduce two measures μ and $\tilde{\mu}$ on $\mathbb{Z}$ such that $\mu Q = \tilde{\mu} Q = 0$, where Q is the matrix of intensity of the pure jump Markov process X.

(6) Describe the two processes in duality with X with respect to the measures μ and $\tilde{\mu}$. Observe these are different processes, and it is possible that one of these processes explodes and not the other.

(7) Use the processes we introduced to construct:
 (a) A $CTMC$ that explodes but has an (infinite) invariant measure.
 (b) A $CTMC$ that has a finite measure μ satisfying $\mu Q = 0$, but which is not an invariant measure.

Exercice 2 — *Harris explosion criterium for branching processes.*
We consider $(X_t)_{t\geq 0}$ stochastic process with values in $\mathbb{N} = \{0, 1, \ldots\}$, with 0 as absorbing state, and counting the population at time t, for a model of population where:

- initially, there is 1 individual, so $X_0 = 1$.
- each individual, independently of others, dies after a time which is exponential with parameter 1, and then gives rise to a random number Z of children, distributed according to ν some probability distribution on $\{0, 2, 3, \ldots\}$.

(1) Show X is a CTMC and provide its matrix of intensity.

(2) Show the jump process associated with X is a random walk with jumps distributed as $Z - 1$, stopped when hitting 0. Deduce that there is a.s. extinction of the population if $\mathbb{E}[Z] \leq 1$.

From now on, we suppose $\mathbb{E}[Z] \in (1, +\infty]$. We let h denote the generating function of Z and f_t that of X_t, defined by

$$h(r) = \mathbb{E}[r^Z], \qquad 0 \leq r \leq 1,$$
$$f_t(r) = \mathbb{E}[r^{X_t}], \qquad 0 \leq r \leq 1,$$

with of course $r^{+\infty} = 0$ in the definition of $f_t(r)$. For $r = 1$, we take the convention

$$f_t(1) = \mathbb{P}(X_t < +\infty) = \lim_{r\to 1, r<1} f_t(r).$$

(3) Show the existence of $q < 1$ such that we have $h(r) < r$ for all r in $[q, 1)$.

We fix q as in last question, and aim to show Harris criterium, which states that there is explosion of the process, namely $f_t(1) < 1$, iff the function $1/(u - h(u))$ is integrable in the neighbourhood of 1, namely

$$(*) \qquad \int_q^1 \frac{1}{u - h(u)} \mathrm{d}u < +\infty$$

(4) We suppose $r \in [0,1)$. Show $t \mapsto f_t(r)$ is derivable at 0, with derivative

$$\frac{\partial}{\partial t} f_t(r)|_{t=0} = h(r) - r.$$

(5) Show we have

$$f_{s+t}(r) = f_s(f_t(r)), \quad \forall s, t \geq 0,$$

and deduce we also have

$$\frac{\partial}{\partial t} f_t(r) = h(f_t(r)) - f_t(r) \quad \forall t \geq 0.$$

(6) We suppose $r \in (q, 1)$. Show we can take $t > 0$ small enough so that the function $s \mapsto f_s(r)$ is decreasing and lower bounded by q on $[0, t]$. For such t, use a change of variable to show

$$\int_{f_t(r)}^{r} \frac{1}{u - h(u)} \mathrm{d}u = t.$$

(7) For r and t as in question (6), deduce that there is no explosion before time t if $(*)$ is not satisfied (namely if the integral is infinite).

(8) We suppose $(*)$ is satisfied (the integral is finite). For r and t as in question (6), show the process explodes, with probability of explosion characterized by $f_t(1) > \rho$, where $\rho := \sup\{r \in [0,1), h(\rho) \geq \rho\}$ and

$$\int_{f_t(1)}^{1} \frac{1}{u - h(u)} \mathrm{d}u = t.$$

(9) We admit the result of last question holds for arbitrary t. What does the probability of explosion converge to? How does it compare to the survival probability of the population?