PARTIAL EXAM – *Solutions*.

Tuesday, February 28. 10.15am-12.15pm.

Usual notations: CTMC for continuous-time Markov chain or equivalently PJMP for Pure Jump Markov process.
The two exercices are independent.

Exercice 1 — *Explosion for the accelerated biased random walk, and duality.*
We consider $(Y_n)_{n\geq 0}$ the biased random walk with positive drift, which is the discrete-time Markov chain with jump distribution $p\delta_1 + q\delta_{-1}$, where $q = 1-p$ is assumed to be strictly smaller than p, or equivalently $p > 1/2$. We recall that this random walk is transient, and $\mathbb{P}(Y_n \to +\infty) = 1$. As usually, we suppose that under the probability measure $\mathbb{P}_0$, this biased random walk is started from 0.

(1) Recall briefly why we have

$$\mathbb{P}_0(T_{-1} < +\infty) = \frac{q}{p}, \quad \mathbb{P}_0(H_0 < +\infty) = 2q,$$

where T_{-1} is the first hitting time of -1, and H_0 the first return time to 0, defined as $\inf\{n \geq 1, Y_n = 0\}$.

Advice: do not hesitate to admit the first statement or even the two if needed, it will not prevent you from answering the following questions.

Sol: We have that $M_n = (q/p)^{Y_n}$ is a nonnegative martingale, therefore $M_{n\wedge T_{-1}}$ is a bounded martingale, which starts from 0 and converges a.s to $M_\infty = \frac{p}{q}\mathbb{1}_{T_{-1}<+\infty}$. By optional stopping, we thus have $\mathbb{E}[M_\infty] = M_0 = 1$, whence $\mathbb{P}_0(T_{-1} < +\infty) = \frac{q}{p}$. For the second point, use Markov property at time 1 to obtain

$$\begin{aligned}\mathbb{P}_0(H_0 < +\infty) &= p\,\mathbb{P}_1(T_0 < +\infty) + q\,\mathbb{P}_{-1}(T_0 < +\infty)\\ &= p.\frac{q}{p} + q.1 = 2q.\end{aligned}$$

(2) Show the Green function of this process is equal to

$$G_Y(0,n) = \begin{cases} \frac{1}{1-2q} & \text{if } n \geq 0,\\ \left(\frac{q}{p}\right)^{|n|}\frac{1}{1-2q} & \text{if } n < 0.\end{cases}$$

Sol: We know that for a discrete time Markov chain,

$$G_Y(0,0) = \frac{1}{1-\mathbb{P}_0(H_0 < +\infty)} = \frac{1}{1-2q}.$$

Moreover, for all $n \in \mathbb{Z}$, we also have

$$G_Y(0,n) = \mathbb{P}_0(T_n < +\infty)G(n,n),$$

(a property that follows easily from Markov property at time T_n.) But by translation invariance, we also have $G_Y(n,n) = G_Y(0,0) = \frac{1}{1-2q}$, and the result of the probability of the hitting times being finite of first question generalizes easily to

$$\mathbb{P}_0(T_n < +\infty) = \begin{cases} 1 & \text{if } n \geq 0, \\ \left(\frac{q}{p}\right)^{|n|} & \text{if } n < 0. \end{cases}$$

The result follows.

We consider now $(q_n)_{n\in\mathbb{Z}} \in [1,+\infty)^{\mathbb{Z}}$ an arbitrary sequence of real numbers larger than or equal to 1 and indexed by $\mathbb{Z}$, and $(X_t)_{t\geq 0}$ the accelerated biased random walk. It is the pure jump Markov process with associated jump process the biased random walk Y we introduced before, and whose waiting time at state n is distributed as an exponential random variable with parameter q_n.

(3) Show that the process (X_t) does not explode if $\sum_{n\geq 0} \frac{1}{q_n} = +\infty$.
Sol: We know that there is a.s. no explosion on the event $\{\sum \frac{1}{q_{Y_n}} = +\infty\}$. But this is an a.s. event under the hypothesis $\sum_{n\geq 0} q_n = +\infty$, as we have Y_n tends a.s. to $+\infty$.

(4) Letting ζ be the explosion time and $G(\cdot,\cdot)$ the Green function of the CTMC X, show we have

$$\mathbb{E}_0[\zeta] = \sum_{n\in\mathbb{Z}} G(0,n).$$

Compute this sum, and deduce that the process explodes if $\sum_{n\geq 0} \frac{1}{q_n} < +\infty$.
Sol: We have

$$\begin{aligned} \zeta &= \sum_{k\geq 0}\sum_{n\in\mathbb{Z}} \mathbb{1}_{Y_k=n}(J_{k+1}-J_k) \\ &= \sum_{n\in\mathbb{Z}}\sum_{k\geq 0} \mathbb{1}_{Y_k=n}(J_{k+1}-J_k) \\ &= \sum_{n\in\mathbb{Z}} \int_0^{+\infty} \mathbb{1}_{X_t=n}\,\mathrm{d}t. \end{aligned}$$

Taking expectation, we obtain $\mathbb{E}_0[\zeta] = \sum_{n\in\mathbb{Z}} G(0,n)$. Now, we have $G(0,n) = \frac{1}{q_n}G_Y(0,n)$, whence

$$\mathbb{E}_0[\zeta] = \frac{1}{1-2q}\sum_{n<0}\left(\frac{q}{p}\right)^{|n|}\frac{1}{q_n} + \frac{1}{1-2q}\sum_{n\geq 0} 1q_n.$$

The first sum is finite as $q < p$ and $q_n \geq 1$, and the second sum is finite by hypothesis. Hence $\mathbb{E}_0[\zeta] < +\infty$, and there is a.s. explosion.

(5) Show the measures ν and $\tilde{\nu}$ on $\mathbb{Z}$ defined by

$$\nu_n = 1, \qquad \tilde{\nu}_n = \left(\frac{p}{q}\right)^n,$$

are invariant for Y, and deduce two measures μ and $\tilde{\mu}$ on $\mathbb{Z}$ such that $\mu Q = \tilde{\mu} Q = 0$, where Q is the matrix of intensity of the pure jump Markov process X.

Sol: The measure ν is invariant: letting K be the transition matrix of Y, we have, for all $n \geq 0$:

$$(\nu K)_n = \nu_{n-1} p + \nu_{n+1} q = p + q = 1 = \nu_n.$$

(We also could argue that the counting measure is always an invariant measure for a random walk.) The measure $\tilde{\nu}$ is reversible:

$$\forall n, \tilde{\nu}_n K_{n,n+1} = \left(\frac{p}{q}\right)^n p = \left(\frac{p}{q}\right)^{n+1} q = \nu_{n+1} K_{n+1,n},$$

and thus invariant. We deduce that the measures μ and $\tilde{\mu}$ satisfy $\mu Q = \tilde{\mu} Q = 0$, where

$$\mu_n = \frac{\nu_n}{q_n} = \frac{1}{q_n},$$
$$\tilde{\mu}_n = \frac{\tilde{\nu}_n}{q_n} = \left(\frac{p}{q}\right)^n \frac{1}{q_n}.$$

(6) Describe the two processes in duality with X with respect to the measures μ and $\tilde{\mu}$. Observe these are different processes, and it is possible that one of these processes explodes and not the other.

Sol: The first dual process $\hat{X}$ has a matrix of intensity $\hat{Q}$ satisfying, for all $n \in \mathbb{Z}$ $\hat{Q}_{n,n} = Q_{n,n} = q_n$,

$$\hat{Q}_{n,n+1} = \frac{\mu_{n+1}}{\mu_n} \hat{Q}_{n+1,n} = \frac{q_n}{q_{n+1}} . (q_{n+1} q) = q_n q,$$

and similarly

$$\hat{Q}_{n,n-1} = q_n p.$$

We recognize the matrix of intensity of a biased random walk, but with negative drift. In particular, $\hat{X}$ is transient to $-\infty$, and explosive iff $\sum_{n \leq 0} \frac{1}{q_n} < +\infty$. The dual process with respect to $\tilde{\mu}$ is... X itself, which we can see by similar computation or by the observation that $\tilde{\nu}$ reversible. In particular, it is explosive iff $\sum_{n \geq 0} \frac{1}{q_n} < +\infty$. Hence it is possible that one dual process explodes and not the other.

(7) Use the processes we introduced to construct:
  (a) A $CTMC$ that explodes but has an (infinite) invariant measure.
  (b) A $CTMC$ that has a finite measure μ satisfying $\mu Q = 0$, but which is not an invariant measure.

Sol:

(a) *We choose a sequence* (q_n) *such that* $\sum_{n\geq 0}\frac{1}{q_n} < +\infty$ *and* $\sum_{n\leq 0}\frac{1}{q_n} = +\infty$. *We then have that* X *explodes but not* $\tilde{X}$. *As the dual process does not explode, we know that* μ *is an invariant measure.*

(b) *We choose a sequence* (q_n) *such that* $\sum_{n\geq 0}\frac{1}{q_n} < +\infty$ *and* $\sum_{n\leq 0}\frac{1}{q_n} < +\infty$. *Then* μ *is a finite measure satisfying* $\mu Q = 0$ *but not an invariant measure. To see it is not an invariant measure, we may either argue that* $\tilde{X}$ *explodes, or we may observe that a transient process cannot have a finite invariant measure.*

Exercice 2 — *Harris explosion criterium for branching processes.*
We consider $(X_t)_{t\geq 0}$ stochastic process with values in $\mathbb{N} = \{0, 1, \ldots\}$, with 0 as absorbing state, and counting the population at time t, for a model of population where:

- initially, there is 1 individual, so $X_0 = 1$.
- each individual, independently of others, dies after a time which is exponential with parameter 1, and then gives rise to a random number Z of children, distributed according to ν some probability distribution on $\{0, 2, 3, \ldots\}$.

(1) Show X is a CTMC and provide its matrix of intensity.
Sol: By construction, 0 is an absorbing state, and when the process X *is in some other state* $n > 0$, *it will jump after an exponential time with parameter* n *(by death of either of the* n *individuals). Whatever the past of the process and the individual that dies, we will have birth of a random number* Z *of children, so the new population will be distributed as* $n + Z - 1$. *So* X *is a CTMC with matrix of intensity*

$$Q_{n,k} = \begin{cases} -n & \textit{if } n = k \neq 0, \\ n\nu(1+k-n) & \textit{if } n \neq 0 \textit{ and } k \in \{n-1, n+1, n+2, \ldots\}, \\ 0 & \textit{otherwise.} \end{cases}$$

(2) Show the jump process associated with X is a random walk with jumps distributed as $Z - 1$, stopped when hitting 0. Deduce that there is a.s. extinction of the population if $\mathbb{E}[Z] \leq 1$.
Sol: We have already proven that the jump process is a RW with jumps distributed as $Z - 1$ *and stopped when hitting 0. If we consider the unstopped random walk, we know it tends to* $-\infty$ *a.s. if the jumps have negative expectation* $\mathbb{E}[Z-1] < 0$, *and is recurrent if the jumps are centered (and integrable), namely if* $\mathbb{E}[Z-1] = 0$. *In both cases, the RW will a.s. hit 0. This stays true of course if the RW is stopped when hitting 0. Thus there is a.s. extinction.*

From now on, we suppose $\mathbb{E}[Z] \in (1, +\infty]$. We let h denote the generating function of Z and f_t that of X_t, defined by

$$h(r) = \mathbb{E}[r^Z], \qquad 0 \leq r \leq 1,$$
$$f_t(r) = \mathbb{E}[r^{X_t}], \qquad 0 \leq r \leq 1,$$

with of course $r^{+\infty} = 0$ in the definition of $f_t(r)$. For $r = 1$, we take the convention

$$f_t(1) = \mathbb{P}(X_t < +\infty) = \lim_{r \to 1, r<1} f_t(r).$$

(3) Show the existence of $q < 1$ such that we have $h(r) < r$ for all r in $[q, 1)$.
Sol: We have $h(1) = 1$ and the leftderivative at 1 is $h'(1) = \mathbb{E}[Z] > 1$ (possibly an infinite derivative). Thus we have $h(r) < r$ for r in some left-neighbourhood of 1.

We fix q as in last question, and aim to show Harris criterium, which states that there is explosion of the process, namely $f_t(1) < 1$, iff the function $1/(u - h(u))$ is integrable in the neighbourhood of 1, namely

$$(*) \qquad \int_q^1 \frac{1}{u - h(u)} \mathrm{d}u < +\infty$$

(4) We suppose $r \in [0, 1)$. Show $t \mapsto f_t(r)$ is derivable at 0, with derivative

$$\frac{\partial}{\partial t} f_t(r)|_{t=0} = h(r) - r.$$

Sol: We have

$$f_t(r) = \sum_j P_{1,j}(t) r^j.$$

We know from Kolmogorov backward equations that $P_{1,j}$ is derivable with derivative

$$P'_{1,j}(t) = \sum_k Q_{1,k} P_{k,j}(t) = \sum_{k \neq 1} Q_{1,k} P_{k,j}(t) - q(1) P_{1,j}(t).$$

Thus

$$\begin{aligned} \sum_j \left| P'_{1,j}(t) r^j \right| &\leq \sum_{k \neq 1} Q_{1,k} \sum_j P_{k,j}(t) + q(1) \sum_j P_{k,j}(t) \\ &\leq 2q(1). \end{aligned}$$

Hence we can take the derivative under the sum to deduce, at time 0,

$$\frac{\partial}{\partial t} f_t(r)|_{t=0} = \sum_j Q_{1,j} r^j = -r + \sum_{j \neq 1} Q_{1,j} r^j = -r + h(r).$$

(5) Show we have

$$f_{s+t}(r) = f_s(f_t(r)), \quad \forall s, t \geq 0,$$

and deduce we also have

$$\frac{\partial}{\partial t} f_t(r) = h(f_t(r)) - f_t(r) \quad \forall t \geq 0.$$

Sol: The descendants of each individual are independent and have the same law. In this sense, we have a branching process. In particlar, starting from k individuals, the process at time t is a sum of k independent copies of X_t, and

$$\mathbb{E}_k[r^{X_t}] = \mathbb{E}[r^{X_t}]^k = f_t(r)^k.$$

Thus, using Markov property at time s:

$$f_{s+t}(r) = \mathbb{E}\left[r^{X_{s+t}}\middle|X_s\right] = \mathbb{E}[f_t(r)^X_s] = f_s(f_t(r)).$$

Taking the derivative in s at time 0, we obtain

$$\frac{\partial}{\partial t} f_t(r) = h(f_t(r)) - f_t(r).$$

(6) We suppose $r \in (q, 1)$. Show we can take $t > 0$ small enough so that the function $s \mapsto f_s(r)$ is decreasing and lower bounded by q on $[0, t]$. For such t, use a change of variable to show

$$\int_{f_t(r)}^{r} \frac{1}{u - h(u)} \mathrm{d}u = t.$$

Sol: For $r \in (q, 1)$ fixed, we write g for the function $g(t) = f_t(r)$. The function g is continuous so we can choose $t > 0$ small so that g stays above q on $[0, t]$. By Question (5), we also have g derivable on $[0, t]$ with $g'(s) = h(g(s)) - g(s)$, which is negative as $g(s) \geq q$. Now we can use the change of variable $u = f_s(r)$ to obtain

$$\int_{f_t(r)}^{r} \frac{1}{u - h(u)} \mathrm{d}u = \int_{t}^{0} -\mathrm{d}s = t.$$

(7) For r and t as in question (6), deduce that there is no explosion before time t if $(*)$ is not satisfied (namely if the integral is infinite).
Sol: Fix r and t as in question (6). We then have for all $r' \in [r, 1]$ and $s \in [0, t]$,

$$f_s(r') \geq f_s(r) \geq q.$$

Thus, for all $r' \in [r, 1]$,

$$\int_{f_t(r')}^{r'} \frac{1}{u - h(u)} \mathrm{d}u = t.$$

We now let r' grow to 1 and $f_t(r')$ grow to $f_t(1)$. If the integral $(*)$ is infinite, we must necessarily have $f_t(1) = 1$ (otherwise we would have $t = \int_{f_t(1)}^{1} \frac{1}{u-h(u)} \mathrm{d}u = +\infty$).

(8) We suppose $(*)$ is satisfied (the integral is finite). For r and t as in question (6), show the process explodes, with probability of explosion characterized by $f_t(1) > \rho$, where $\rho := \sup\{r \in [0, 1), h(\rho) \geq \rho\}$ and

$$\int_{f_t(1)}^{1} \frac{1}{u - h(u)} \mathrm{d}u = t.$$

Sol: If the integral $(*)$ is finite, by the same argument we must have

$$\int_{f_t(1)}^{1} \frac{1}{u - h(u)} \mathrm{d}u = t, \tag{1}$$

which implies in particular $f_t(1) < 1$, thus the probability of explosion before time t is positive. Moreover $f_t(1)$ is in $(\rho, 1]$ and is characterized by (1), because the function

$$p \mapsto \int_p^1 \frac{1}{u - h(u)} \mathrm{d}u$$

is strictly decreasing on $(\rho, 1]$.

(9) We admit the result of last question holds for arbitrary t. What does the probability of explosion converge to? How does it compare to the survival probability of the population?

Sol: We note that $\rho < 1$ is the largest (and actually the only) solution on $[0, 1)$ to $h(\rho) = \rho$. It is a classical result that ρ is also the extinction probability of the Galton-Watson process with child distribution ν... and also clearly the extinction probability of the current population model.

By the definition of ρ, we have $h(r) < r$ on $(\rho, 1)$, and we must have

$$\int_\rho^1 \frac{1}{u - h(u)} \mathrm{d}u = +\infty,$$

either by a simple study of the integral with the observation $h'(\rho) < 1$, or because there cannot be explosion and extinction of the population, and thus $f_t(1)$ has to always stay larger than ρ, and thus for arbitrary t,

$$\int_\rho^1 \frac{1}{u - h(u)} \mathrm{d}u \geq \int_{f_t(1)}^1 \frac{1}{u - h(u)} \mathrm{d}u = t.$$

Finally the fact that

$$\int_{f_t(1)}^1 \frac{1}{u - h(u)} \mathrm{d}u \to +\infty$$

implies that we must have $f_t(1) \to \rho$, namely the probability of explosion by time t tends to the survival probability. In other words, there is a.s. explosion on the survival event.