Exercise 1. — *Martingales associated with a Gaussian integral.*

We say that a random process $(X_t)_{t\geq 0}$ adapted to a filtration $\mathcal{F}_t$ has independent increments if and only if for any $s, t \geq 0$, the random variable $X_{t+s} - X_t$ is independent of $\mathcal{F}_t$.

1. Let X be a random process with independent increments. Show that the following random processes are $(\mathcal{F}_t)$-martingales, provided that the expectations are well-defined.
   (a) $(X_t - \mathbb{E}[X_t])_{t\geq 0}$
   (b) $((X_t - \mathbb{E}[X_t])^2 - \mathbb{E}[(X_t - \mathbb{E}[X_t])^2])_{t\geq 0}$
   (c) $(e^{\lambda X_t}/\mathbb{E}[e^{\lambda X_t}])_{t\geq 0}$, where $\lambda \in \mathbb{R}$ is a parameter.
2. Let $f \in L^2_{\text{loc}}(\mathbb{R}_+)$ and $X_t = G(f\mathbf{1}_{[0;t]}) =: \int_0^t f(s)dB_s$. Show that the random processes X, $(X_t^2 - \int_0^t f^2(s)ds)_{t\geq 0}$, and $(e^{\lambda X_t - \frac{\lambda^2}{2}\int_0^t f^2(s)ds})_{t\geq 0}$, are martingales.

Exercise 2. — *The Brownian bridge.*

Let $(B_t)_{t\in[0,1]}$ be a standard Brownian motion. For $x \in \mathbb{R}$, define the process $(X_t^x)_{t\in[0,1]}$ by

$$X_t^x = B_t - t(B_1 - x),$$

and let $\mathbb{P}_x$ be its law on $\mathcal{C}([0,1], \mathbb{R})$

1. Show X^x is a Gaussian process, and compute its mean as well as its covariance function.
2. Prove that $\mathbb{P}_x$ is a version of the conditional law of the Wiener measure W on $C([0,1], \mathbb{R})$ knowing $B_1 = x$. In other words, as B_1 has law $\mathcal{N}(0,1)$, this means proving that for any measurable set $A \in C([0,1], \mathbb{R})$, we have

$$W(A) = \int_{\mathbb{R}} \mathbb{P}_x(A) \frac{1}{\sqrt{2}\pi} e^{-\frac{x^2}{2}} dx.$$

Exercise 3. — *Hitting times of the Brownian motion.*

Let B be a real-valued $(\mathcal{F}_t)$-Brownian motion starting from 0. For any $x \in \mathbb{R}$, we let T_x be the hitting time of x by B, that is $T_x := \inf\{t \geq 0, B_t = x\}$.

1. Using an appropriate martingale, find for every $a < 0 < b$ the probability $\mathbb{P}(T_a < T_b)$.
2. Using an appropriate martingale, find for every $x \in \mathbb{R}$ the Laplace transform of T_x.
3. Show that the random process $(T_x)_{x\geq 0}$ has independent and stationary increments, that is for any $0 \leq a \leq b$, the random variable $T_b - T_a$ is independent of $\sigma(T_c, 0 \leq c \leq a)$ and has same law as T_{b-a}.
4. Find for every $a < 0 < b$ the Laplace transform of $\min(T_a, T_b)$.
   Indication : you may use the martingale $(cosh(\lambda(B_t - (a+b)/2))e^{-\lambda^2 t/2})_{t\geq 0}$ (and show that it is a martingale indeed).

Exercise 4. — *About the quadratic variation.*

Let B be a standard Brownian motion starting from 0.

1. Let $(\Delta_n)_{n\geq 0} = (t_0^n, \dots, t_{k_n}^n)_{n\geq 0}$ be a sequence of subdivisions of $[0,1]$ with mesh decreasing towards 0, and nested, that is for any n we have $\{t_0^n, \dots, t_{k_n}^n\} \subset \{t_0^{n+1}, \dots, t_{k_{n+1}}^{n+1}\}$. Show that we have
$$\sum_{i=1}^{k_n} \left(B_{t_i^n} - B_{t_{i-1}^n}\right)^2 \xrightarrow{a.s.} 1.$$
It may be useful to introduce an appropriate backwards martingale.
2. Without using the nesting hypothesis, show that for some sequences (Δ_n) of subdivisions of $[0,1]$ with mesh decreasing towards 0, we have that almost surely,
$$\limsup_{n\to\infty} \sum_{i=1}^{k_n} \left(B_{t_i^n} - B_{t_{i-1}^n}\right)^2 = +\infty.$$

Exercise 5. — *A differentiable Gaussian process.*

Let $(X_t)_{t\in[0;1]}$ be a centred Gaussian process. We assume that the application $(t,\omega) \mapsto X_t(\omega)$ is measurable from $[0;1]\times\Omega$ to $\mathbb{R}$. We denote by K the covariance function of X.

1. Show that the mapping $t \mapsto X_t$ is continuous from $[0;1]$ to $L^2(\Omega)$ if and only if K is continuous on $[0;1]^2$. We assume in the sequel that this condition is satisfied.
2. Let $h : [0;1] \to \mathbb{R}$ be a measurable function such that $\int_0^1 |h(t)|\sqrt{K(t,t)}dt < \infty$.
   (a) Show that a.s. the integral $\int_0^1 h(t)X_t dt$ is absolutely convergent.
   (b) We let
   $$Z = \int_0^1 h(t)X_t dt \quad \text{and} \quad \forall n \geq 1, \quad Z_n := \sum_{i=1}^{n} X_{\frac{i}{n}} \int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t)dt.$$
   Assuming that $\int_0^1 |h(t)|dt < \infty$, show that $(Z_n)_{n\geq 1}$ converges towards Z in $L^2(\Omega)$. Deduce that Z is a Gaussian random variable.
3. Assume that K is $\mathcal{C}^2$. Show that for any $t \in [0;1]$, the limit
$$\dot{X}_t := \lim_{s\to t} \frac{X_s - X_t}{s-t}$$
exists in $L^2(\Omega)$. Show that $(\dot{X}_t)_{t\in[0;1]}$ is a centred Gaussian process. Compute its covariance function.

Properties of the Brownian motion

In what follows, we let $(B_t)_{t\geq 0}$ be a standard Brownian motion starting from 0. We also let $S_t := \sup_{0\leq s\leq t} B_s$ for $t \geq 0$.

Exercise 6. — *Time inversion.*

1. Show that the process $(W_t)_{t\geq 0}$ defined by $W_0 = 0$ and for any $t > 0$ by $W_t = tB_{1/t}$ is indistinguishable from a real-valued Brownian motion starting from 0 (you may check first that W is an almost-Brownian motion).
2. Deduce that
$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \quad \text{a.s.}$$

Exercise 7. — *Non-differentiability.*

Using a $0-1$ law, show that almost surely,

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = +\infty, \quad \liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

Deduce that for any $t \geq 0$, almost surely the function $s \mapsto B_s$ is not right differentiable in t.

Exercise 8. — *Local Maxima.*

Show that a.s. the local maxima of the Brownian motion are distinct, i.e. almost surely for any rational numbers $p < q < r < s$, we have

$$\sup_{p\leq t\leq q} B_t \neq \sup_{r\leq t\leq s} B_t.$$

Exercise 9. — *Zeros of the Brownian motion.*

Let $H := \{t \in [0;1] : B_t = 0\}$ be the zeros set of the Brownian motion in $[0;1]$. Show that a.s. H is compact and has null Lebesgue measure, and using the strong Markov property, show that a.s. H has no isolated point.

Exercise 10. — *Return times.*

Let $S := \inf\{t \geq 0 : B_t = 1\}$ and $T := \inf\{t \geq S : B_t = 0\}$.

1. Show that these random variables are finite a.s. (as by convention, $\inf \emptyset = +\infty$).
2. Is the random variable T a stopping time ?
3. Give the law of T.

Exercise 11. — *The arcsine law.*

We let $T := \inf\{t \geq 0 : B_t = S_1\}$.

1. Show that $T < 1$ a.s. and then show that T is not a stopping time.
2. Show that the three random variables S_t, $S_t - B_t$ and $|B_t|$ have same law.
3. Show that T has an arcsine distribution, the arcsine distribution being a probability law with density f defined for any $t \in \mathbb{R}$ by

$$f(t) = \frac{1}{\pi\sqrt{t(1-t)}} \mathbf{1}_{\{]0;1[\}}(t).$$

4. Show that the results of questions 1 and 3 remain true if we replace T by $L := \sup\{t \leq 1 : B_t = 0\}$.

Exercise 12. — *Law of the iterated logarithm.*

The aim of this exercise is to show the following property of the Brownian motion :

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log(t)}} = 1 \quad a.s.$$

We let $h(t) := \sqrt{2t \log\log(t)}$ for any $t > 0$.

1. Show that for any $t > 0$,

$$\mathbb{P}(S_t > u\sqrt{t}) \sim_{u\to\infty} \frac{2\mathrm{e}^{-u^2/2}}{u\sqrt{2\pi}}$$

2. Let $r, c \in \mathbb{R}$ such that $1 < r < c^2$. Study the behaviour of $\mathbb{P}(S_{r^n} > ch(r^{n-1}))$ as $n \to \infty$ and deduce that almost surely,

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log(t)}} \leq 1.$$

3. Show that almost surely, there are infinitely many n such that

$$B_{r^n} - B_{r^{n-1}} \geq \sqrt{\frac{r-1}{r}} h(r^n).$$

   Conclude on the announced result.

Let us now derive a corollary from this result.

4. Compute $\liminf_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log(t)}}$.
5. For any $s \geq 0$, show that a.s.

$$\limsup_{t \downarrow 0} \frac{B_{t+s} - B_s}{\sqrt{2t \log \log(t)}} = 1 \quad \text{and} \quad \liminf_{t \downarrow 0} \frac{B_{t+s} - B_s}{\sqrt{2t \log \log(t)}} = -1.$$

6. Deduce that a.s. the trajectories of the Brownian motion are nowhere 1/2-Hölder continuous.