Exercise 1. — *Random scaling of a local martingale.*

Let M be a local martingale, and U be an $\mathcal{F}_0$-measurable random variable. Show that the process $(N_t)_{t\geq 0} := (UM_t)_{t\geq 0}$ is a local martingale.

Exercise 2. — *Bracket of a Gaussian process.*

Let $(M_t)_{t\geq 0}$ be a (true) martingale with continuous paths starting from $M_0 = 0$. We assume that $(M_t)_{t\geq 0}$ is a Gaussian process.

1. Show that for any $t \geq 0$ and $s > 0$, the random variable $M_{t+s} - M_t$ is independent of $\sigma(M_r, 0 \leq r \leq t)$.
2. Show that there exists a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that a.s. $\forall t \leq 0$, $\langle M, M\rangle_t = f(t)$.

Exercise 3. — *Mean formula.*

Let B be an $\mathcal{F}_t$-Brownian motion starting from 0, and H an adapted process with continuous paths. Show that $\frac{1}{B_t}\int_0^t H_s dB_s$ converges in probability when $t \downarrow 0$ towards a limit to be determined.

Exercise 4. — *Paths of M and paths of $\langle M\rangle$, part 1.*

Let M be a local martingale starting from 0. For $n \geq 0$, we set

$$T_n := \inf\{t \geq 0, |M_t| = n\}, \qquad U_n := \inf\{t \geq 0, \langle M\rangle_t = n\}.$$

Prove that
- $\langle M\rangle_\infty$ is a.s. finite. on $\{T_n = +\infty\}$.
- M has an almost sure limit $M_\infty \in \mathbb{R}$ on $\{U_n = +\infty\}$.

Deduce that the sets $\{\langle M\rangle_\infty < +\infty\}$ and $\{(M_t)_{t\geq 0}$ has a finite limit$\}$ coincide, up to a negligible set.

Exercise 5. — *Paths of M and paths of $\langle M\rangle$, part 2.*

We show in this exercise that the constancy intervals of a local martingale M and of $\langle M\rangle$ are a.s. the same. By continuity of these processes and usual reasoning, it is enough to show that for every fixed interval $[a, b]$, we have almost surely

$$(M \text{ is constant on } [a, b]) \Leftrightarrow \langle M\rangle_a = \langle M\rangle_b.$$

1. Using the approximations of $\langle M\rangle_b - \langle M\rangle_a$ associated to a subdivision of $[a, b]$ to show one of the implication is. Why doesn't it give the other implication ?
2. We introduce the local martingale N defined for any $t \geq 0$ by

   $$N_t := M_{a+t} - M_a,$$

   and adapted to the filtration $(\mathcal{F}_{t+a})_{t\geq 0}$. For $\varepsilon > 0$, we also introduce

   $$T_\varepsilon := \inf\{t \geq 0, \langle N\rangle_t = \varepsilon\}.$$

   Show that for $t \geq 0$, we have $\mathbb{E}[N^2_{t\wedge T_\varepsilon}] \leq \varepsilon$, and deduce that N_t is a.s. equal to 0 on $\{\langle M\rangle_{a+t} = \langle M\rangle_a\}$. Conclude.

Exercise 6. — *Uniform convergence of local martingales*

For any $n \geq 1$, we let $M^n = (M_t^n)_{t\geq 0}$ be a local martingale starting from 0. We assume in this exercise that

$$\lim_{n\to\infty} \langle M^n, M^n \rangle_\infty = 0$$

in probability.

1. Let $\varepsilon > 0$ and, for any $n \geq 1$,

$$T_\varepsilon^n := \inf\{t \geq 0 : \langle M^n, M^n \rangle_t \geq \varepsilon\}.$$

   Show that T_ε^n is a stopping time, and show that the stopped local martingale

$$M_t^{n,\varepsilon} = M_{t\wedge T_\varepsilon^n}^n, \quad t \geq 0,$$

   is a martingale bounded in L^2.

2. Show that

$$\mathbb{E}\Big[\sup_{t\geq 0} |M_t^{n,\varepsilon}|^2\Big] \leq 4\varepsilon^2.$$

3. Noticing that for any $a \geq 0$,

$$\mathbb{P}\Big(\sup_{t\geq 0} |M_t^n| \geq a\Big) \leq \mathbb{P}\Big(\sup_{t\geq 0} |M_t^{n,\varepsilon}| \geq a\Big) + \mathbb{P}(T_\varepsilon^n < \infty),$$

   show that

$$\lim_{n\to\infty} \Big(\sup_{t\geq 0} |M_t^n|\Big) = 0$$

   in probability.

Exercise 7. — *Brownian motion in* $\mathbb{R}^d$.

In this exercise, we suppose $d \geq 2$ and let $B_t = (B_t^1, \dots, B_t^d)$ be a BM in $\mathbb{R}^d$, starting from $x = (x_1, \dots x_d) \neq 0$.

1. Show that $|B_t|^2$ is a continuous semi-martingale with decomposition

$$|B_t|^2 = |x|^2 + M_t + dt,$$

   for a local martingale M that we shall specify.

2. Show that M is a true martingale and $M_t = 2\int_0^t |B_s| d\tilde{B}_s$, for some Brownian motion $\tilde{B}$ that we shall specify.
   As a consequence, $X_t = |B_t|^2$ is a solution of the SDE

$$dX_t = 2\sqrt{X_t}\; d\tilde{B}_t + dt.$$

   Such a solution is also called a squared Bessel process of dimension d.

3. For $0 < r < |x|$, define $T_r := \inf\{t \geq 0, |B_t| = r\}$. We let f_d be the function defined for any $y \geq 0$ by $f_d(y) := \log y$ if $d = 2$ and $f_d(y) := y^{2-d}$ if $d \geq 3$.
   Show that $f_d(|B_{t\wedge T_r}|)_{t\geq 0}$ is a martingale,

4. Show that for $0 < r < |x| < R$, we have

$$\mathbb{P}(T_r < T_R) = \frac{f(R) - F(|x|)}{f(R) - f(r)}.$$

   Deduce that almost surely,

(a) The Brownian motion B does not hit $0 \in \mathbb{R}^d$.

(b) If $d \geq 3$, then $|B_t| \to \infty$ when $t \to \infty$.

(c) If $d = 2$, the path $\{B_t, t \geq 0\}$ is dense in $\mathbb{R}^2$.

5. In dimension 3, show that $(|B_t|^{-1})_{t\geq 0}$ is a local martingale bounded in L^2, but not a true martingale.

Exercise 8. — *Tanaka's formula.*

Let B be a Brownian motion starting from 0.

1. For any $\varepsilon \in (0,1]$, we let $g_\varepsilon(x) = \sqrt{\varepsilon + x^2}$ (for $x \in \mathbb{R}$). Show that
$$g_\varepsilon(B_t) = \sqrt{\varepsilon} + M_t^\varepsilon + A_t^\varepsilon,$$
where M^ε is a continuous martingale in L^2 starting from 0, and A^ε is an increasing process starting from 0 to be determined.
2. For $x \in \mathbb{R}$ we let $\mathrm{sgn}(x) := \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$. Show that the process M defined for any $t \geq 0$ by
$$M_t := \int_0^t \mathrm{sgn}(B_s) dB_s,$$
is a Brownian motion, and show that for all $t \geq 0$, the random variable M_t^ε tends to M_t in L^2.
3. For $t \geq 0$, we let $A_t := |B_t| - M_t$. Show that for any $t \geq 0$, the random variable A_t^ε tends to A_t in L^2, and deduce that A is (indistinguishable from) an increasing process.
4. Show that A is almost surely constant on the open intervals included in the set $\{t \geq 0, B_t \neq 0\}$.
Hint : you may start by using the convergence of A_t^ε towards A_t in order to deduce that for $\eta > 0$ and $s < t$, we have a.s.
$$(A_t - A_s)\mathbf{1}_{\{\forall r \in [s,t], |B_r| \geq \eta\}} = 0.$$
5. For $t \geq 0$, we let $R_t := \sup\{r \leq t, B_r = 0\}$ be the last time at which B is in 0 before t. Show that a.s.,
$$A_t = A_{R_t} = -M_{R_t} = \sup\{-M_r, r \leq t\},$$
and deduce the law of A_t.

The formula
$$|B_t| = M_t + \sup\{-M_r, r \leq t\},$$
where $M_t = \int_0^t sgn(B_s)dB_s$, is called Tanaka's formula. For $t \geq 0$, the r.v. A_t is called the local time of B in 0 at time t, and "measures" the set $\{s \in [0,t], B_s = 0\}$. But this heuristic description is not rigorous, as this set has Hausdorff dimension 1/2, so its Lebesgue measure is 0. However, it is possible to choose an adequate gauge function, such that A_t coincides with the Hausdorff measure of this set for this gauge function.