Exercise 1. — *Lévy's area.*

Let $(X_t, Y_t)_{t\geq 0}$ be a two-dimensional Brownian motion, starting from 0 (we recall that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are then two independent 1-dimensional Brownian motions). We let for any $t \geq 0$,

$$S_t := \int_0^t X_s dY_s - \int_0^t Y_s dX_s. \quad \textit{(Lévy's area).}$$

1. Compute $\langle S, S\rangle_t$, and deduce that $(S_t)_{t\geq 0}$ is a (true) martingale that is square-integrable (that is $\mathbb{E}[S_t^2] < \infty$ for all $t \geq 0$).
2. Let $\lambda > 0$. Show the following equality :

$$\mathbb{E}[\mathrm{e}^{i\lambda S_t}] = \mathbb{E}[\cos(\lambda S_t)].$$

3. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞ function. Using Itô's formula, give the explicit decomposition of the semimartingales

$$\begin{aligned} Z_t &= \cos(\lambda S_t), \\ W_t &= -\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t) \end{aligned}$$

as a sum of a local martingale and a finite variation process (you may start by writing the decomposition of $X_t^2 + Y_t^2$). Check that $\langle Z, W_t\rangle_t = 0$

4. Using Itô's formula again, show that if f satisfies the differential equation

$$f''(t) = f'(t)^2 - \lambda^2,$$

then $Z_t \mathrm{e}^{W_t}$ is a local martingale.

5. Let $r > 0$. Show that the function

$$f(t) = -\ln(\mathrm{ch}(\lambda(r-t)))$$

is a solution of the differential equation given in the previous question. Deduce the formula

$$\mathbb{E}[\mathrm{e}^{i\lambda S_r}] = \frac{1}{\mathrm{ch}(\lambda r)}.$$

Exercise 2. — *Geometric Brownian Motion.*

Let us consider the SDE

$$dX_t = \sigma X_t dB_t + \lambda X_t dt,$$

with $\sigma > 0$ and $\lambda \in \mathbb{R}$. Show the strong existence and uniqueness of solutions, the solution for a given initial condition X_0 being given by

$$X_t = X_0 \exp\left(\sigma B_t + (\lambda - \frac{\sigma^2}{2})t\right).$$

This process is called the geometric Brownian motion. Notice that it remains positive. It is used as a simple model for price evolution of stocks (Black and Scholes model), in particular because the inflation rates $(X_{t+s} - X_s)/X_s$ are independent on disjoint time intervals.

Exercise 3. — *SDE with a time-change.*

We consider the Stochastic Differential Equation

$$dX_t = \sigma(X_t)dB_t,$$

where $\sigma : \mathbb{R} \to K$ is continuous and K is a compact set in $]0, +\infty[$. We aim to show weak existence and uniqueness of solutions to this equation, where the solution is given by the time-change of a Brownian motion.

1. Suppose that X is a solution starting from x, and define

$$A(t) = \int_0^t \sigma^2(X_s)ds.$$

   Show that A is invertible, with

$$A^{-1}(t) = \int_0^t \sigma^{-2}(X_{A^{-1}(s)})ds.$$

2. We let

$$\tilde{B}_t := X_{A^{-1}(t)}.$$

   Show that $\tilde{B}$ is a Brownian Motion, and that $X_t = \tilde{B}_{\inf\left\{s\geq 0, \int_0^s \sigma^{-2}\left(\tilde{B}_r\right)dr > s\right\}}$.

3. Conclude.

Exercise 4. — *A time-homogeneous SDE.*

We consider the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt,$$

where σ and b are continuous and bounded, and such that $\sigma \geq \varepsilon$ for some $\varepsilon > 0$ and $\int_{\mathbb{R}} |b(x)|dx < \infty$.

1. Suppose that there exists a solution X. Show that there exists F a C^2 (strictly) increasing function such that $F(X_t)$ is a martingale. Give an expression of F only depending on σ and b.
2. Show that $Y_t = F(X_t)$ is a solution of the SDE

$$dY_t = \sigma'(Y_t)dB_t,$$

   where σ' is a function to be determined. Deduce from last exercise the weak existence and uniqueness of solutions to the SDE satisfied by X.