ENS de Lyon - M2 **Tuesday, December 18, 2018**

Stochastic calculus exam (3h)

Lecture notes allowed. Answers in french or in english are accepted. Approximate scale : Ex 1 on 10 points. Ex 2 on 7 points. Problem on 14 points.

In the whole exam, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, on which we consider continuous and adapted processes.

We use the notation $a \wedge b = \min(a, b)$. When M is a local martingale, we denote by $\langle M \rangle$ its quadratic variation process, and by $\mathcal{E}(M)$ its associated exponential local martingale, defined by

$$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}.$$

We provide the following generalized version of Dubins-Schwarz theorem : *Suppose M is a local martingale started from 0. Then there exists a brownian motion B (possibly defined on an enlarged probability space and not adapted to the same filtration as M), such that for every $t \geq 0$, we have*

$$M_t = B_{\langle M \rangle_t}.$$

EXERCISE I : A nonnegative semimartingale with a positive bias.

We consider B a standard Brownian motion, $\lambda > 0$ some fixed parameter, and X a continuous semimartingale. For $x \geq 0$, we denote by

$$T_x := \inf\{t \geq 0, X_t = x\}$$

the hitting time of x by the semimartingale X. We further suppose that X satisfies

$$X_t = 1 + B_{t\wedge T_0} + \int_0^{t\wedge T_0} \frac{\lambda}{X_s} \mathrm{d}s.$$

In particular, X solves the SDE

$$\begin{cases} X_0 &= 1 \\ \mathrm{d}X_t &= \mathbb{1}_{X_t>0}\mathrm{d}B_t + \mathbb{1}_{X_t>0}\frac{\lambda}{X_t}\mathrm{d}t. \end{cases}$$

1. Show that for $b > 1$, we have $T_0 \wedge T_b$ is almost surely finite.

2. For $\alpha \in \mathbb{R}$ and $a \in (0,1)$, we consider $F_{\alpha,a}$ a C^2 function that coincides with $x \mapsto x^\alpha$ on $[a,+\infty)$. Write down Itō formula for the semimartingale $F_{\alpha,a}(X_t)$.
3. We suppose $\lambda \neq 1/2$. For a parameter $\alpha = \alpha(\lambda)$ that we will determine, deduce that the process $(X^\alpha_{t\wedge T_a})_{t\geq 0}$ is a local martingale, for any $a \in (0,1)$.
4. For $a \in (0,1)$ and $b > 1$, compute $\mathbb{P}(T_a < T_b)$.
5. In this question only, we suppose $\lambda > 1/2$. Show X_t tends to $+\infty$ a.s. as $t \to +\infty$. (One may observe that $(X^\alpha_{t\wedge T_a})_{t\geq 0}$ is a bounded martingale).
6. In this question only, we suppose $\lambda < 1/2$. Show that T_0 is a.s. finite.
7. Finally, we suppose $\lambda = 1/2$. Proceed similarly as in the previous questions to show that a.s., the hittings times T_x are finite for all $x \in (0,+\infty)$. In other words, the process X_t is recurrent on $(0,+\infty)$.

EXERCISE II : Hitting time of the sphere for the biased planar Brownian motion.

We consider $B_t = (B^{(1)}_t, B^{(2)}_t)_{t\geq 0}$ a planar Brownian motion started from 0, and T the hitting time of the sphere $\mathbb{S} = \{(x,y), x^2+y^2 = 1\}$.

1. Explain briefly why $(B^{(1)}_T, B^{(2)}_T)$ is independent from T and uniform on the sphere.
2. Introducing an appropriate martingale, deduce that the Laplace transform of T is given for $\lambda \geq 0$ by
$$\mathbb{E}[e^{-\lambda T}] = \frac{1}{\phi\left(\sqrt{2\lambda}\right)},$$
where ϕ is the function defined for $x \geq 0$ by
$$\phi(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos t}\mathrm{d}t.$$
Note you are not asked to compute this integral.
3. For a parameter $c \in \mathbb{R}$, we introduce now $C_t = (C^{(1)}_t, C^{(2)}_t)_{t\geq 0}$ the Brownian motion with drift ct, defined for $t \geq 0$ by $C^{(1)}_t = B^{(1)}_t + ct$ and $C^{(2)}_t = B^{(2)}_t$. We also let T_C be the hitting time of the sphere for the process C. Show that, for any F measurable and bounded functional on the Wiener space $C(\mathbb{R}_+)$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$, we have
$$\mathbb{E}\left[F((C_{t\wedge T_C})_{t\geq 0})\right] = \mathbb{E}\left[F((B_{t\wedge T})_{t\geq 0})e^{cB_T - \frac{c^2}{2}T}\right].$$
4. Deduce the Laplace transform of T_C is given by
$$\mathbb{E}[e^{-\lambda T_C}] = \frac{\phi(c)}{\phi\left(\sqrt{2\lambda + c^2}\right)}.$$

PROBLEM : Local martingales with product of finite variation

This problem starts with two preliminary questions, and then studies the local martingales whose product is a process of finite variation.

1. In this question, we consider Y a stochastic process with continuous paths started from $Y_0 = 1$, and staying null after its first hitting time of 0. Thus, writing $T_0 := \inf\{t \geq 0, Y_t = 0\}$, we have $Y_t = 0$ for every $t \geq T_0$. We further suppose that there exists a nondecreasing sequence of stopping times $(S_n)_{n\geq 0}$ converging to T_0 and such that for every $n \geq 0$, the stochastic process $(Y_{t\wedge S_n})_{t\geq 0}$ is a local martingale. Prove Y is a local martingale.
   Hint : You may first suppose that the process Y is bounded.
2. Suppose M is a local martingale started from 0 such that $\langle M\rangle_\infty$ is a.s. finite. Show
   (a) The probability of the event $\{\langle M\rangle_\infty \leq u, M_\infty + \frac{1}{2}\langle M\rangle_\infty \geq v\}$ goes to 0 when $v \to \infty$, uniformly on the chosen local martingale M.
   (b) The probability of the event $\{M_\infty + \frac{1}{2}\langle M\rangle_\infty \leq u, \langle M\rangle_\infty \geq v\}$ goes to 0 when $v \to \infty$, uniformly on the chosen local martingale M.
3. We now consider (X, Y) a pair of local martingales and suppose their product $Z_t = X_t Y_t$ is a process of finite variation.
   (a) Show the local martingale N defined by
   $$N_t = \int_0^t X_s \mathrm{d}Y_s + \int_0^t Y_s \mathrm{d}X_s$$
   is indistinguishable from 0.
   (b) Compute $\langle N\rangle$ and deduce that the process
   $$t \mapsto \int_0^t Z_s \mathrm{d}\langle X, Y\rangle_s$$
   is nonincreasing.
   (c) Deduce that Z_t^2 is nonincreasing.

   As a consequence, if $X_0 = 0$, then for every $t \geq 0$, we have $Z_0 = 0$ and thus either $X_t = 0$ or $Y_t = 0$. In the following, we suppose $X_0 = Y_0 = 1$ and the processes X and Y stay still after their first hitting time of 0.
4. In this question only, we further suppose that X and Y stay (strictly) positive. Show there is a local martingale L starting from 0 such that $X_t = \mathcal{E}(L)_t$ and $Y_t = \mathcal{E}(-L)_t$. Reciprocally, for every local martingale L starting from 0, show the product of the local martingales $\mathcal{E}(L)$ and $\mathcal{E}(-L)$ is indeed a process of finite variation.
5. For $\varepsilon \in (0, 1)$, we let $T_\varepsilon := \inf\{t \geq 0, X_t = \varepsilon \text{ or } Y_t = \varepsilon\}$ and define
   $$L_t^\varepsilon := \int_0^{t\wedge T_\varepsilon} \frac{1}{X_s} \mathrm{d}X_s.$$

(a) For $\varepsilon \in (0,1)$ and $t \geq 0$, show we have $X_{t\wedge T_\varepsilon} = \mathcal{E}(L^\varepsilon)_t$ and $Y_{t\wedge T_\varepsilon} = \mathcal{E}(-L^\varepsilon)_t$.

(b) We also let $T_0 := \inf\{t \geq 0, X_t = 0 \text{ or } Y_t = 0\}$ and $T_0(X) := \inf\{t \geq 0, X_t = 0\}$. On the event $T_0 = T_0(X) < +\infty$, show that $\langle L^\varepsilon\rangle_{+\infty}$ tends to $+\infty$ a.s. when ε decreases to 0.

(c) Deduce that, on the event $T_0 < +\infty$, we have $X_{T_0} = Y_{T_0} = 0$ a.s.

6. In this question, suppose X is any local martingale starting from 1 and staying null after its first hitting time of 0. Deduce from previous questions that there is a unique local martingale Y starting from 1 and staying null after its first hitting time of 0 such that XY is a process of finite variation, and propose a description of it.