ENS de Lyon - M2 **January 4, 2017**

Stochastic calculus, final exam (3 hrs)

The notes you have taken during the class are authorized. Other documents are not.

Exercise 1. An SDE on [0,1)

We consider the following stochastic differential equation on the time interval $[0,1)$:

$$dX_t = dB_t - \frac{X_t}{1-t}dt, \tag{1}$$

1. If one is given a solution of (1), write a SDE that the process Y defined by $Y_t = \frac{X_t}{1-t}$ must satisfy. Deduce that for any initial condition X_0, there is a unique (up to indistinguishability) strong solution to (1), given by

$$X_t = (1-t)X_0 + (1-t)\int_0^t \frac{1}{1-s}dB_s.$$

In the following, we consider such a solution X, and keep the notation $Y_t = \frac{X_t}{1-t}$.

2. For $t \geq 0$, we let

$$\tilde{B}_t := Y_{t/(1+t)} - Y_0.$$

Show $\tilde{B}$ is a Brownian motion. Deduce that X_t converges almost surely to 0 when t goes to 1, and then that it is indistinguishable from a continuous semimartingale which is still a strong solution to (1), but which converges *everywhere* to 0 when t goes to 1.

We still call $(X_t)_{t\in[0,1]}$ that continuous semimartingale, extended by continuity in 1 by putting $X_1 = 0$. We further suppose $X_0 = 0$.

3. Show $(X_t)_{t\in[0,1]}$ is a centered gaussian process, with covariance function given by

$$\mathrm{Cov}(X_s, X_t) = s(1-t), \qquad 0 \leq s \leq t \leq 1.$$

Do you recognize a process we met during the class ?

4. We now consider on $[0,1)$ a slightly different SDE :

$$dX_t = dB_t + \frac{X_t}{1-t}dt, \tag{2}$$

With similar ways as question 1, solve this equation with initial condition $X_0 = 0$, and discuss the behavior of its solution when t tends to 1.

Exercise 2. Characterisation of a local martingale and of its quadratic variation

1. Suppose a is a function of finite variation on $[0, t]$ and f a bounded measurable function on $[0, t]$ with values in $\mathbb{R}\backslash\{0\}$, such that

$$\forall s \in [0, t], \qquad \int_0^s f(r) da(r) = 0,$$

(the integral above is a Stieltjes integral). Show a is constant.

2. Suppose M is a local martingale, V is a process of finite variation and their product $(M_t V_t)_{t \geq 0}$ is a local martingale. Show that the process $t \mapsto \int_0^t M_s dV_s$ is indistinguishable from 0.
3. Suppose X is a (continuous) local martingale and A is a (continuous and adapted) process of finite variation starting from 0. Under the hypothesis that $\exp(X - \frac{1}{2}A)$ is a local martingale, show that $A = \langle X \rangle$.
4. We remove the assumption that X is a local martingale, and suppose instead it is a continuous and adapted process. Under the hypothesis that the processes $Z^{(\lambda)}$ defined for $\lambda \in \mathbb{R}$ by

$$Z_t^{(\lambda)} = \exp(\lambda X_t - \frac{\lambda^2}{2} A_t), \qquad t \geq 0$$

are local martingales, show X is a local martingale and $A = \langle X \rangle$.

Indication : We may use the sequence of stopping times

$$T_n := \inf\{t \geq 0, |X_t| \geq n \textit{ or } |A_t| \geq n\}$$

to reduce the local martingales, and introduce a sequence of local martingales that converge pointwise to X.

Exercise 3. Tanaka formula and local time

In this exercise, B is a brownian motion starting from 0.

1. Suppose $(f_\varepsilon)_{\varepsilon > 0}$ is a family of uniformly bounded measurable functions from $\mathbb{R}$ to $\mathbb{R}$, that converge pointwise to f, necessarily measurable and bounded. Show that the processes $f(B)$ and $f(B_\varepsilon)$ are in $L^2(B)$ (where $L^2(B)$ is the standard notation used in class), and for each $t \geq 0$, we have

$$\int_0^t f_\varepsilon(B_s) dB_s \underset{\varepsilon \to 0}{\to} \int_0^t f(B_s) dB_s$$

in L^2.

2. For $\varepsilon \in (0,1]$, define $g_\varepsilon(x) = \sqrt{\varepsilon + x^2}$. Show we have

$$g_\varepsilon(B_t) = \sqrt{\varepsilon} + M_t^\varepsilon + A_t^\varepsilon,$$

where M^ε is a square integrable continuous martingale (starting from 0) and A^ε an increasing process (starting from 0) that we shall determine.

3. We let $sgn(x) := \mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x<0\}}$, and for $t \geq 0$,

$$M_t := \int_0^t sgn(B_s) dB_s$$

Show M is a Brownian motion, and, for every $t \geq 0$, M_t^ε converges to M_t in L^2.

4. For $t \geq 0$, we let $A_t := |B_t| - M_t$. Show that for $t \geq 0$, the r.v. A_t^ε converges to A_t in L^2, and deduce that A is (indistinguishable from) a continuous increasing process.

5. Show A is almost surely constant on all open subsets of $\{t \geq 0, B_t \neq 0\}$.
Indication : We may first use convergence of A_t^ε to A_t to show that for $\eta > 0$ and $s < t$, we have

$$(A_t - A_s)\mathbb{1}_{\{\forall r \in [s,t], |B_r| \geq \eta\}} = 0 \qquad a.s.$$

6. For $t \geq 0$, we let $R_t := \sup\{r \leq t, B_r = 0\}$ be the last time the Brownian motion B hits 0 before time t. Show we have a.s.

$$A_t = A_{R_t} = -M_{R_t} = \sup\{-M_r, r \leq t\},$$

and deduce the law of A_t.

We have proven

$$|B_t| = M_t + \sup\{-M_r, r \leq t\}$$

with $M_t = \int_0^t sgn(B_s)dB_s$, a formula known as Tanaka formula. For $t \geq 0$, the r.v. A_t is also known as the local time at level 0 of the Brownian motion B on time interval $[0,t]$. The remainder of this exercise deepens the study of local times.

7. Writing $(B_t)_+ := 0 \vee B_t$, show we also have

$$(B_t)_+ = \int_0^t \mathbb{1}_{\{B_s \geq 0\}} dB_s + \frac{1}{2}A_t.$$

8. For $x \in \mathbb{R}$, we define the martingale

$$N_t^x := \int_0^t \mathbb{1}_{\{B_s \geq x\}} dB_s.$$

Show that for every $x < y$ and $t \geq 0$, we have

$$\begin{aligned} E\left[(\langle N^x - N^y \rangle_t)^2\right] &= 2\int_0^t \int_0^s \mathbb{P}(B_r \in [x,y))\mathbb{P}(B_s - B_r) \in [x,y))drds \\ &\leq c(t)(y-x)^2, \end{aligned}$$

where $c(t)$ is a finite constant depending only on t.

9. We now fix $t > 0$. Bound the fourth moment of $N_t^x - N_t^y$ and deduce there is a continuous version of the process indexed by $\mathbb{R}$

$$x \mapsto N_t^x$$

We now consider this continuous version, and define

$$L_x := 2\left((B_t - x)_+ - (-x)_+ - N_t^x\right),$$

so that in particular $L_0 = A_t$. Note that L_x depends clearly on t, but we do not specify this dependance in the notation (recall $t > 0$ is now fixed)

10. Show that for any continuous function f with compact support, we can interchange Lebesgue and Itô integrals to get

$$\int_{-\infty}^{+\infty} f(x) N_t^x dx = \int_0^t \left(\int_{-\infty}^{+\infty} f(x) \mathbb{1}_{\{B_s \geq x\}} dx \right) dB_s.$$

We may think about a Riemann approximation of Lebesgue integrals, and use again question 1 of this exercise.

11. Show that for any continuous function f with compact support, we have

$$\int_{-\infty}^{+\infty} f(x) L_x dx = \int_0^t f(B_s) ds.$$

To prove this, you may introduce the function

$$F(x) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(z) dz dy = \int_{-\infty}^{+\infty} f(z)(x-z)_+ dz.$$

and observe $F'(x) = \int_{-\infty}^{+\infty} f(z) \mathbb{1}_{x \geq z} dz$ and $F''(x) = f(x)$.