M2A stochastic calculus : Partial exam (2h)

No documents allowed. A precise and concise writing is expected.

EXERCISE I

The aim of this exercise is to prove the following theorem :

Theorem 1. *Let M be a (continuous) local martingale starting from 0. Almost surely, on any nontrivial interval $[0,t]$ with $t>0$, the function $s \mapsto M_s$ is either constantly equal to 0, or of infinite variation.*

1. Explain briefly the difference with the somehow similar theorem given in the lecture.

2. Suppose X is an adapted stochastic process with leftcontinuous paths with values in a topological space, endowed with its Borel $\sigma-$field. Show the hitting times of open sets are optional times.

3. Suppose X is an adapted stochastic process with continuous paths, and V is its total variation process, defined by

$$V_t = \int_0^t |dX_s|.$$

Show V is adapted and has leftcontinuous paths with values in $\mathbb{R}_+ \cup \{+\infty\}$. We consider here the Alexandrov compactification of $\mathbb{R}_+$, so the complementary of the open sets containing $+\infty$ are the compact subsets of $\mathbb{R}_+$.

4. Suppose M is a (continuous) local martingale starting from 0, as in the statement of the theorem, and $(\mathcal{F}_t)_{t\geq 0}$ is its canonical filtration. Show M is also a $(\mathcal{F}_{t+})_{t\geq 0}-$local martingale.
Hint : You may first suppose that M is a true martingale with continuous paths.

5. For $r>0$, let $T_r := \inf\{t \geq 0, \int_0^t |dM_s| > r\}$. For $t>0$, show we have

$$\mathbb{E}[(M_{t\wedge T_r})^2] = 0.$$

6. Conclude.

EXERCISE II

In this exercise, we consider B a brownian motion starting from 0 and we fix three parameters $a, b > 0$ and $m \in \mathbb{R}$. We define the process X by $X_t = B_t - mt$ for $t \geq 0$, as well as the hitting times

$$\begin{aligned} T_{-a} &:= \inf\{t \geq 0, X_t = -a\}, \\ T_b &:= \inf\{t \geq 0, X_t = b\}, \\ T &:= \min(T_{-a}, T_b). \end{aligned}$$

1. Show that T is a.s. finite, and prove, for $\lambda \geq 0$, the equality

$$\mathbb{E}[e^{-\lambda T} \mathbb{1}_{T_b < T_{-a}}] = \frac{e^{-\rho_- a} - e^{-\rho_+ a}}{e^{\rho_+ b - \rho_- a} - e^{\rho_- b - \rho_+ a}},$$

where ρ_+ (resp. ρ_-) is the nonnegative (resp. nonpositive) solution of the equation

$$\rho^2 - 2m\rho - 2\lambda = 0.$$

2. Deduce the probability of the events $T_b < T_{-a}$ and $T_b < +\infty$.
3. Compute also, in the case $m = 0$:

$$\mathbb{E}[T \mathbb{1}_{T_b < T_{-a}}] = \frac{ab(a + 2b)}{3(a + b)}.$$

Hint : You may first show

$$\mathbb{E}\left[e^{-\frac{\lambda^2}{2} T} \mathbb{1}_{T_b < T_{-a}}\right] = \frac{\sinh(\lambda a)}{\sinh(\lambda(a + b))}.$$

EXERCISE III

The aim of this exercise is to define and study the fractional Brownian motion. We let $p \in (0, 1/2)$ and $\alpha = 1 - 2p$.

1. For every $t \geq 0$, show the function

$$f_t : x \mapsto |x - t|^{-p} - |x|^{-p}$$

is in $L^2(\mathbb{R})$.

2. Writing $\| \cdot \|$ the $L^2(\mathbb{R})$-norm, show we have $\|f_t\|^2 = t^\alpha \|f_1\|^2$ and $\|f_t - f_s\|^2 = |t - s|^\alpha \|f_1\|^2$ for $s, t \geq 0$.

3. Let G be a gaussian measure with intensity the Lebesgue measure on $\mathbb{R}$ and $X_t := G(f_t)$, for $t \geq 0$. Show X is a centered gaussian process with covariance function

$$\mathbb{E}[X_s X_t] = c(t^\alpha + s^\alpha - |t-s|^\alpha),$$

   for some $c > 0$.

   The process X is called fractional Brownian motion with Hurst exponent α.

4. Show X is α-autosimilar, in the sense that for $\lambda > 0$, the processes $(X_{\lambda t})_{t\geq 0}$ and $(\lambda^\alpha X_t)_{t\geq 0}$ have the same finite-dimensional distributions.

5. Show X has a continuous modification, and even a locally β-Hölder continuous modification, for parameters β to be determined.