ENS de Lyon - M2 **November 7, 2016**

Stochastic calculus, partial exam (2 hrs)

The notes you have taken during the class are authorized. Other documents are not.

The aim of this partial exam is to prove and understand the Doob-Meyer decomposition theorem of a submartingale as the sum of a martingale and an increasing process.

In the Part I, we work in discrete-time indexed by $\mathbb{N} = \{0, 1, 2, \ldots\}$ with a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. In Parts II and III, we work in continuous time indexed by $[0, 1]$, with a filtration $(\mathcal{F}_t)_{t\in[0,1]}$. We use the same notation for these filtrations as there shall be no confusion. We suppose these filtrations **satisfy the usual conditions.**

All the processes we introduce are real-valued, start from 0, and are adapted and càdlàg. When you introduce a new process, you may have to check it satisfies these properties...

PART I : discrete-time

A process $X = (X_n)_{n\in\mathbb{N}}$ is called **integrable** if the random variables X_n are in L^1, **predictable** if for all $n \geq 1$, the random variable X_n is $\mathcal{F}_{n-1}$-measurable.

Theorem (Doob decomposition theorem).
An integrable process $X = (X_n)_{n\in\mathbb{N}}$ has a unique decomposition as $X = M + A$ with
. $(M_n)_{n\in\mathbb{N}}$ *martingale.*
. $(A_n)_{n\in\mathbb{N}}$ *integrable and predictable.*

As mentioned in the introduction, we implicitly ask that X, A and M are adapted and $X_0 = A_0 = M_0 = 0$.

1. Suppose $X = M + A$ is such a decomposition. Show that for $n \geq 1$, we must have

$$A_n - A_{n-1} = \mathbb{E}[X_n|\mathcal{F}_{n-1}] - X_{n-1}.$$

2. Prove Doob theorem, with the unique decomposition (M, A) given for $n \geq 1$ by

$$\begin{aligned} A_n &= \sum_{k=1}^{n} (\mathbb{E}[X_k|\mathcal{F}_{k-1}] - X_{k-1}) \\ M_n &= X_n - A_n. \end{aligned}$$

3. Show X is a submartingale iff A is an increasing process.

4. Give an example of a submartingale X which can be decomposed in two different ways $X = M+A = M'+A'$, with M and M' martingales and A and A' increasing, where A and A' are not asked to be predictable (but still adapted).

Indication : It suffices to consider one time-step, with processes indexed by $\{0,1\}$.

PART II : continuous-time

In this part, we will use the notion of **weak-L^1 convergence**, which we now define.

Definition. *We say a sequence* (Y_n) *of random variables in* L^1 *converges weakly in* L^1 *to* $Y \in L^1$*, and we write* $Y_n \underset{n\to\infty}{\overset{w\,L^1}{\longrightarrow}} Y$ *if for every bounded random variable* Z*, we have*

$$\mathbb{E}[Y_n Z] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[YZ]$$

We admit the following theorem, due to Dunford and Pettis :

Theorem (Dunford-Pettis theorem)**.** *If* $(Y_n)_{n\in\mathbb{N}}$ *is a uniformly integrable sequence of random variables, then there exists a subsequence which converges weakly in* L^1*.*

1. Suppose $Y_n \overset{w-L^1}{\longrightarrow} Y$. Show that for any σ-field $\mathcal{G}$, we have $\mathbb{E}[Y_n|\mathcal{G}] \overset{w-L^1}{\longrightarrow} \mathbb{E}[Y|\mathcal{G}]$.

We call $\mathcal{T}$ the set of stopping times relative to the filtration $(\mathcal{F}_t)_{t\in[0,1]}$, and with values in $[0,1]$. The second important definition of this part is the following :

Definition. *We say an (adapted and càdlàg) process* $X = (X_t)_{t\in[0,1]}$ *is* ***of class*** D *if the family of random variables* $(X_T)_{T\in\mathcal{T}}$ *is uniformly integrable.*

Note this implies in particular that $(X_t)_{t\in[0,1]}$ is uniformly integrable.

2. Show a martingale M is always of class D. Similarly, show an integrable increasing process A is always of class D.

It follows that the sum of a martingale and an integrable increasing process is always a submartingale of class D. Doob-Meyer theorem shows in particular that the converse is true :

Theorem (Doob-Meyer decomposition theorem, part A)**.**
A submartingale X *of class* D *has a decomposition as* $X = M + A$ *with*
. M *a martingale*
. A *an integrable increasing process.*

Here again, we stress that X, M and A are càdlàg, adapted, and $X_0 = M_0 = A_0 = 0$.

The idea to prove this theorem is to approximate M and A with discrete-time processes, using Doob decomposition theorem. For $n \geq 0$, we let $D_n = \{k2^{-n}, k = 0, 1, \dots 2^n\}$. By Doob decomposition theorem, we can write, for $t \in D_n$, $X_t = M_t^n + A_t^n$, where $(M_t^n)_{t\in D_n}$ is a discrete martingale indexed by time in D_n and $(A_t^n)_{t\in D_n}$ an increasing process such that A_t^n is $\mathcal{F}_{t-2^{-n}}$-measurable.

For $n \geq 0$ and $\lambda \in [0, +\infty)$, we let $T^n_\lambda = \inf\{t \in D_n, A^n_{t+2^{-n}} > \lambda\}$, with the convention $\inf \emptyset = 1$.

3. Show T^n_λ is a stopping time and we have
$$\mathbb{E}[A^n_1 \mathbb{1}_{A^n_1 > 2\lambda}] \leq 2\mathbb{E}[A^n_1 - A^n_{T^n_\lambda}] \leq 2\mathbb{E}[(X_1 - X_{T^n_\lambda})\mathbb{1}_{A^n_1 > \lambda}].$$
4. Show that $\mathbb{P}(A^n_1 > \lambda)$ goes to 0 as λ goes to $+\infty$ **uniformly in** n and deduce that $(A^n_1)_{n\geq 0}$ is uniformly integrable.

Using Dunford-Pettis theorem, we define A_1 as the weak-L^1 limit of some subsequence of $(A^n_1)_{n\geq 0}$. We also define $M_1 = X_1 - A_1$, then $M_t = \mathbb{E}[M_1|\mathcal{F}_t]$ and then $A_t = X_t - M_t$.

5. Justify that the processes $(M_t)_{t\in[0,1]}$ and $(A_t)_{t\in[0,1]}$ can be defined this way, as adapted and càdlàg processes starting from 0.
6. Finish the proof of Doob-Meyer decomposition theorem, part A.

PART III : Uniqueness/Naturalness

In this part, we still write A for the increasing process constructed in Part II.

1. Let $(Z_t)_{t\in(0,1]}$ be a bounded càdlàg martingale, not necessarily[1] starting from 0. Prove that, for $n \geq 0$, we have
$$\mathbb{E}[Z_1 A_1] = \mathbb{E}[\sum_{k=1}^{2^n} Z_{k2^{-n}}(A_{k2^{-n}} - A_{(k-1)2^{-n}})].$$
Indication : Use $\mathbb{E}[Z_{k2^{-n}}|\mathcal{F}_{(k-1)2^{-n}}] = Z_{(k-1)2^{-n}}$.
2. Deduce that we also have
$$\mathbb{E}[Z_1 A_1] = \mathbb{E}[\int_{(0,1]} Z_t dA_t].$$
where the right hand side is a Stieltjes integral.

Indication : Write the sum of last question as $\int Z^n_t dA_t$ *for some well-defined* Z^n*...*

To get the uniqueness in the Doob-Meyer decomposition theorem, we need the property of naturalness, which can be seen as a continuous-time analogue to predictability.

Definition. *An increasing process* $(A'_t)_{t\in[0,1]}$ *is called* ***natural*** *if for every bounded càdlàg martingale* $(Z_t)_{t\in[0,1]}$*, we have*
$$\mathbb{E}[Z_1 A'_1] = \mathbb{E}[\int_{(0,1]} Z_{t-} dA'_t].$$

Theorem (Doob-Meyer decomposition theorem, part B).
The process A is natural. The Doob-Meyer decomposition $X = M + A$ with M martingale and A integrable, increasing, and natural, is unique (up to indistinguishability).

1. In this part, Z will always denote a càdlàg martingale which does not have to start from 0

3. Let Z be a càdlàg bounded martingale. Prove

$$\mathbb{E}[\int_{(0,1]} Z_{t-} dA_t] = \lim_{n\to\infty} \mathbb{E}[\sum_{k=1}^{2^n} Z_{(k-1)2^{-n}}(A_{k2^{-n}} - A_{(k-1)2^{-n}})],$$

and then

$$\begin{aligned}\mathbb{E}[Z_{(k-1)2^{-n}}(A_{k2^{-n}} - A_{(k-1)2^{-n}})] &= \mathbb{E}[Z_{(k-1)2^{-n}}(A^n_{k2^{-n}} - A^n_{(k-1)2^{-n}})] \\ &= \mathbb{E}[Z_{k2^{-n}}(A^n_{k2^{-n}} - A^n_{(k-1)2^{-n}})].\end{aligned}$$

4. Deduce from previous questions and the construction of A, that this process is natural.
5. Suppose $X = M' + A'$ is another decomposition, with M' martingale and A' integrable, increasing, and natural. Show that we have

$$\mathbb{E}[\int_{(0,1]} Z_{t-} dA_t] = \mathbb{E}[\int_{(0,1]} Z_{t-} dA'_t],$$

6. Deduce that we have

$$\mathbb{E}[Z_1 A_1] = \mathbb{E}[Z_1 A'_1],$$

for every càdlàg martingale $(Z_t)_{t\in[0,1]}$. Conclude that A_1 and A'_1 are equal almost surely, and then A and A' are indistinguishable.