A new proof for a decreasing dimension theorem

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1 Introduction

This note provides a new proof to a conjecture formulated by I. Rival which concerns the dimension of partially ordered sets (*posets* for short).

Dimension reducing extension (Rival): Given a poset, is there a way to add one relation at a time so that the dimension never increases on the way to finding a linear extension ?

This question was reported in the list of open problems of J. Spinrad [3] and a first proof was given by M. Pouzet, K. Reuter, I. Rival and N. Zaguia in [1]. Their proof is part of a more general study concerning the extensions of a poset. Our proof is rather presented as a direct modification of a realizer of the poset. These kinds of proofs which directly handle a realizer with elementary transformations (permute some elements in a linear extension, copy, cut, paste some parts of linear extensions) are often used to set results about the dimension of posets (see [5] for examples).

We will use the following notations. Let $P = (X, <_P)$ be a poset (i.e. $<_P$ is a transitive, antisymmetric, irreflexive relation on X). Two different elements x and y of X are *comparable* if $x <_P y$ or $y <_P x$. Otherwise they are *incomparable*. A *linear extension* L of P is a poset which is an extension of P ($\forall x, y \in X$, $x <_P y$ implies $x <_L y$) and a total order ($\forall x, y \in X$ such that $x \neq y$, $x <_L y$ or $y <_L x$). A family of k extensions $L_1, L_2, \ldots, L_k$ of P is called a k-realizer of P if for all x,y in X, $x <_P y$ iff $\forall i,\ 1 \leq i \leq k,\ x <_{L_i} y$. A *critical pair* xy of P is a pair of incomparable elements x and y of P such that for all z in X, $z <_P x$ implies $z <_P y$ and $z >_P y$ implies $z >_P x$. The set of all the critical pairs of P is denoted by $Crit(P)$. A family of linear extensions $L_1, L_2, \ldots, L_k$ *reverse* $Crit(P)$ if for all $xy \in Crit(P)$, there exists a linear extension L_i such that $y <_{L_i} x$. Let x and y be two incomparable elements of P, the notation $P + xy$ represents the poset obtain from P by adding a comparability relation $x < y$ and computing the transitive closure of this new relation. *Adding only one relation* (as in the conjecture) means that the transitive closure does not add some other relations.

2 A new proof

We can now state the two well-known results that will be used for the proof. Proposition 1 is both a definition and a characterization of the *dimension* of a poset (see [2] and [5] for details). Proposition 2 is another way to present critical pairs.

Proposition 1 *Let P be a poset. Then the next three statements are equivalent:*

(i) The dimension $dim(P)$ of P is equal to k.

(ii) The minimum number of linear extensions of a realizer of P is equal to k.

(iii) The minimum number of linear extensions that reverse $Crit(P)$ is equal to k.

Proposition 2 *Let P be a poset. Then xy is a critical pair if and only if $P+xy$ only adds one relation to P.*

The conjecture can be written as below and the proof is given with some remarks.

Proposition 3 (Dimension reducing extension) *Let P be a poset. Then there exists a critical pair xy such that $dim(P+xy) \leq dim(P)$.*

Proof : Consider a k-realizer L_1,L_2, ..., L_k of P, with $k = dim(P)$ (we call this a *minimum realizer* of P). The total order associated to each L_i will be denoted by $<_i$.

The first $k-1$ linear extensions L_1, ..., L_{k-1} don't reverse all the critical pairs of $Crit(P)$ (because of proposition 1). So there exists a critical pair (x, y) which is reversed only by L_k.

As $y <_k x$, consider in L_k the set A of elements between y and x, $A = \{z \in X, y <_k z <_k x\}$. All the elements of A are incomparable to x and y in P: let $z \in A$, three cases may occur,

- $z <_P x$ which implies $z <_P y$ (as (x, y) is a critical pair), but it is in contradiction with $y <_k z$,
- $z >_P y$ which implies $z >_P x$, it is also impossible,
- z is incomparable with x and y, it is the only possible case.

Now consider z the immediate predecessor of x in L_k. z is incomparable to x and y, so two cases occur (see Figures 1 and 2):

- $\forall i, 1 \leq i \leq k-1, x <_i z$. In that case, inverting x and z in L_k adds only one new comparison to P. By proposition 2, it means that (x, z) was a critical pair of P and when you add it to P as a comparable pair, $dim(P + xz) \leq dim(P)$. The proposition is proved.

- $\exists i, 1 \leq i \leq k-1, z <_i x$. In that case, interverting x and z in L_k doesn't add any new comparison to P, and thus we obtain a new k-realizer of P where y and x are closer to each other. Then the preceding arguments apply to this new realizer. So we can invert x with his immediate predecessor, and we repeat this operation until it adds one new comparison to P (at least y would suit).

$\square$

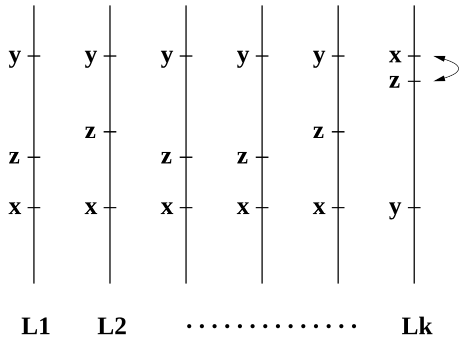

Figure 1: First configuration of the realizer.

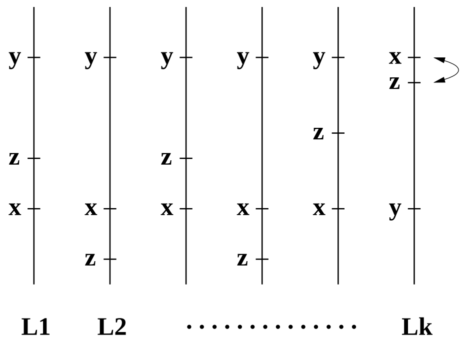

Figure 2: Second configuration of the realizer.

In the proof we first consider a critical pair that is reversed only by L_k and after some inversions of the elements in L_k, we find a critical pair that can be added to P without increasing its dimension. This critical pair is also reversed only by L_k. Note that this reasoning can be done for each linear extension L_i, $1 \leq i \leq k$, and it leads to a different critical pair each time. Thus we have the following corollary.

Corollary 1 *Let P be a poset. Then there exist at least $dim(P)$ different critical pairs such that each one can be added to P without increasing its dimension.*

This is a tight bound in the sense that there exist posets of any dimension verifying $dim(P) = |Crit(P)|$. See for instance the bipartite posets S_n with minimal elements $x_1, x_2, \ldots, x_n$ and maximal elements $y_1, y_2, \ldots, y_n$ such that $x_i <_{S_n} y_j$ if and only if $i \neq j$. For such a poset shown on Figure 3, we have $dim(S_n) = |Crit(S_n)| = n$ (see [5] for a proof).

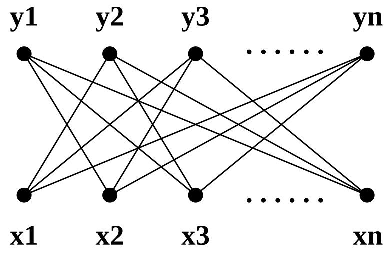

Figure 3: Bipartite poset S_n.

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