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Lattices of tilings and stability

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Abstract

Many tiling spaces such as domino tilings of fixed figures have an underlying lattice structure. This lattice structure corresponds to the dynamics induced by flips. In this paper, we further investigate the properties of these lattices of tilings. In particular, we point out a *stability* property: the set of all the shortest sequences of flips joining to fixed tilings also have a lattice structure close to the lattice of all tilings.

We also show that some of these properties also apply to other discrete dynamical systems and more generally may be satisfied by some partially ordered sets. It gives a new perspective on the lattice structure of tiling spaces and enables to deduce some of their properties only by means of partial order theoretical tools.

Keywords: tilings, lattices, stability of the tilting operation, partial order theory.

Résumé

De nombreux espaces de pavages, tels que les pavages d'une figure par des dominos, peuvent être munis d'une structure de treillis. Cette structure de treillis est induite par des transformations locales élémentaires (*flips*). Dans cet article, nous approfondissons l'étude des propriétés de ces treillis. En particulier, nous mettons en évidence une propriété de stabilité lorsque l'on considère l'ensemble des plus courts chemins reliant deux pavages par des séquences de transformations élémentaires.

Nous montrons aussi que certaines de ces propriété (dont la stabilité) s'appliquent à d'autres systèmes dynamiques discrets et plus généralement à certains ensembles ordonnés. Ces résultats donnent un nouveau point de vue sur la structure de treillis des esapces de pavages et certaines propriétés s'avèrent être des conséquences de théorèmes de théorie des ordres.

Mots-clés: pavages, treillis, stabilité de l'opération de retournement, théorie des ordres.

Lattices of tilings and stability

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Abstract

Many tiling spaces such as domino tilings of fixed figures have an underlying lattice structure. This lattice structure corresponds to the dynamics induced by flips. In this paper, we further investigate the properties of these lattices of tilings. In particular, we point out a *stability* property: the set of all the shortest sequences of flips joining to fixed tilings also have a lattice structure close to the lattice of all tilings.

We also show that some of these properties also apply to other discrete dynamical systems and more generally may be satisfied by some partially ordered sets. It gives a new perspective on the lattice structure of tiling spaces and enables to deduce some of their properties only by means of partial order theoretical tools.

1 Definitions and notations

1.1 Partial orders and lattices

Let $(P, \leq_P)$ be a partial order (order for short) on the ground set P (denoted by P if there is no ambiguity on the relation $\leq_P$). We denote by $|P|$ the cardinal of P. The same order relation $\leq_P$ restricted to a subset of P is called a *suborder* of P. For all $x, y \in P$, the suborder $[x, y] = \{z \in P | x \leq_P z \leq_P y\}$ is called an *interval* of P. The *dual* of P, denoted P^d, is the order on the same ground set obtained by reversing the relation $\leq_P$: for all x, y, we have $x \leq_{P^d} y$ if and only if $y \leq_P x$. Let P and Q be two orders, the *parallel composition* of P and Q is the order on the disjoint union of the ground sets of P and Q such that the induced orders on P and Q remain the same, but an element of P and an element of Q are never comparable. The notation is $P \bigcup Q$.

We also define for $(P, \leq_P)$ its *cover relation* denoted by $\prec_P$ and defined for all $x, y \in P$ by $x \prec_P y$ if $x \neq y$, $x \leq_P y$ and $[x, y] = \{x, y\}$.

A *lattice* $(L, \leq_L)$ is an order such that for all $x, y \in L$, the pair $\{x, y\}$ has an infimum $x \wedge_L y$ and a supremum $x \vee_L y$. If there is no ambiguity, we will just use the notations $\wedge$ and $\vee$. A *meet semilattice* $(L, \leq_L)$ is an order such that for all $x, y \in L$, the pair $\{x, y\}$ has an infimum $x \wedge_L y$. A *distributive lattice* is a lattice such that $\wedge$ is distributive with respect to $\vee$ and inversely. A *sublattice* of L is a suborder S of L which is a lattice and such that for all $x, y \in S$, $x \wedge_S y = x \wedge_L y$ and $x \vee_S y = x \vee_L y$. A subset A of P is called an *ideal* (or *downset*) of P if $x \in A$ and $y \leq_P x$ implies $y \in A$. For instance, for all $x \in P$, $\downarrow_P x = \{y \in P, y \leq_P x\}$ is an ideal of P, which is called the *ideal generated by x in P*. We denote by $I(P)$ the set of all ideals of P ordered by inclusion. $I(P)$ is a distributive lattice, where the supremum of elements is their union, the infimum their intersection. Moreover, from Birkhoff's representation theorem [6, 8], we know that for any distributive lattice L there exists an order P such that L is isomorphic to $I(P)$.

Let L be a lattice. We denote by $G(L)$ the undirected graph obtained from the transitive reduction of L by transforming each directed edge into an undirected edge.

1.2 Tilings

1.2.1 The square grid

Let Λ be the planar grid of the Euclidean plane $\mathbb{R}^2$. A *vertex* of Λ is a point with both integer coordinates. Let $v = (x, y)$ be a vertex of Λ.

A *cell* of Λ is a (closed) unit square whose corners are vertices. Two cells are *4-neighbors* (respectively *8-neighbors*) if they share an edge (respectively (at least) a vertex).

Two vertices of Λ are *neighbors* if they are both ends of a same edge of a cell of Λ. Hence, each vertex v has four neighbors which are canonically called the *East, West, North and South neighbors* of v. An ordered pair of neighbor vertices is called an *arc* or an *edge* of Λ.

We assume that cells of Λ are colored as a checkerboard. By this way, we have black cells and white cells, and two cells sharing an edge have different colors. For each arc (v, v') of Λ, we define the *spin* of (v, v') (denoted by $sp(v, v')$) by :

- $sp(v, v') = 1$ if (v, v') if an ant moving form v to v' has a white cell on its left side (and a black cell on its right side),
- $sp(v, v') = -1$ otherwise.

A *path* of Λ is a sequence $(v_0, v_1, \ldots, v_p)$ of vertices such that for each integer i such that $0 \leq i < p$, v_{i+1} is a neighbor of v_i. This path is a *cycle* if, moreover, $v_p = v_0$. The cycle is *elementary* if $v_i = v_j$ and $i \neq j$ imply that $\{i, j\} = \{0, p\}$. An elementary cycle C divides the cells of the plane into interior cells and exterior cells (according to Jordan's theorem).

We denote by $W(C)$ (respectively $B(C)$) the number of interior white (respectively black) cells according to C.

The elementary cycles can be partitionned according to the orientation : there are the *clockwise* and *counterclockwise* cycles.

Lemma 1 *Let $C = (v_0, v_1, \ldots, v_p)$ be a counterclockwise cycle. We have the equality:*

$$\sum_{i=0}^{p-1} sp(v_i, v_{i+1}) = 4(w(C) - B(C))$$

Proof. Obvious, by induction on the number of enclosed cells (see for example [23] for details). □

1.2.2 Figures

A *figure* F of Λ is a finite union of cells of Λ. The set of edges of F (denoted by E_F) is the set of ordered pairs (v, v') such that the line segment $[v, v']$ is a side of a cell of F.

We state $F = F_1 \cup F_2 \ldots \cup F_p$, where, for each integer i, F_i is a 4-connected component of F. For each connected component F_i, we fix a vertex w_i of its boundary (for example, w_i can be chosen as the leftmost vertex of the lowest vertices of F_i).

The only infinite connected (for the 8-connectivity) component of $\mathbb{R}^2 \setminus F$ is denoted by H_∞. The other ones are called *holes* of F. A connected figure such that H_∞ is the only connected component of $\mathbb{R}^2 \setminus F$ is called a *polygon* of Λ.

A figure is *balanced* if it contains as many black cells as white cells. A figure is *fully balanced* if it is balanced and all its holes are also balanced. In this paper, we are only focused on fully balanced figures. Some extensions of the notions presented in this paper are possible for balanced figures, but this general framework implies technical difficulties (see [20]) which are not interesting for our purpose.

Because of problems due to both types of connectivity for cells, we replace (until the end of the paper) each vertex v of F such that each edge issued from v is in the boundary of F by two vertices v_1 and v_2, each of them being

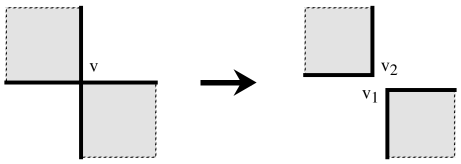


Figure 1: vertex duplication according to 4-connectivity of F and 8-connectivity of $\mathbb{R}^2 \setminus F$.

connected two exactly two neighbors of v (see figure 1). By this way, the contour of each hole is an elementary cycle of F.

A *domino* is a figure formed from two cells sharing an edge, which is called the *central axis* of the domino. A *tiling* T of a figure F is a set of dominoes included in F, with pairwise disjoint interiors (i. e. there is no overlap), such that the union of tiles of T equals F (i. e. there is no gap). Of course, each figure which can be tiled is balanced.

1.2.3 Flips

A *local flip* (see figure 2) is the replacement in T of the pair of dominoes which cover a 2×2 square by the other pair which can cover S. Let v denote the central vertex of the square, a new tiling T_v is obtained by this replacement. We say that T_v is obtained from T by a flip around v.

Two tilings such that on can be obtained from the other one by a single flip are *neighbors*. A *path of tilings* is a sequence $(T_0, T_1, \ldots, T_p)$ of tilings such that for each integer i such that $O \leq i < p$, T_{i+1} and T_i are neighbors. The integer p is the length of the path.

Two tilings, T and T' are *connected by flips* if there exists a path of tilings linking T and T'. in this case, the *flip distance* $d(T, T')$ is the minimal number of successive flips necessary to transform T into T'. A path linking T and T' of length $d(T, T')$ is called a *geodesic*. The *space of tilings* generated by the pair (T, T') is the symmetric graph $G_{(T,T')} = (V_{(T,T')}, E_{(T,T')})$ where $V_{(T,T')}$ is the set of tilings which are on a geodesic linking T and T', and a pair (T_1, T_2) of $(V_{(T,T')})^2$ is in E if T_1 and T_2 are neighbors.

Our goal is the study of the structure of such spaces of tilings.

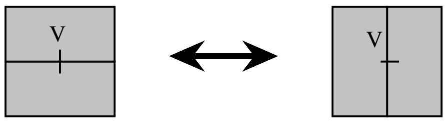


Figure 2: A local flip.

1.2.4 Height functions

Definition 1 *Let T be a tiling of a figure F and (v, v') be an arc of E_F. The T-value (denoted by val_T) is the function form E_F to $\mathbb{Z}$ is defined by :*

- $val_T(v, v') = -3sp(v, v')$ *and there exists a domino of T whose a symmetry axis is* $[v, v']$,
- $val_T(v, v') = sp(v, v')$ *otherwise.*

The function val_T is a tool to encode the tilings : for each pair (T, T') of tilings of F, if $val_T = val_{T'}$, then we have $T = T'$.

Notice that for each arc (v, v') such that $[v, v']$ is on the boundary of F, we necessarily have $val_T(v, v') = sp(v, v')$, thus $val_T(v, v')$ does not depend on T.

Let $(v_0, v_1, \ldots, v_p)$ be a path of the figure F (i. e. for each integer i such that $0 \leq i < p$, (v_i, v_{i+1}) is in E_F) and T be a tiling of F. The height value of this path for T is the sum : $\sum_{i=0}^{p-1} val_T(v_i, v_{i+1})$.

Proposition 1 *Let T be a tiling of a fully balanced figure F. The height value of any cycle of F for T is null.*

This proposition is a generalization of a theorem from J. H. Conway [7] about tilings of polygons.

Proof. (sketch) It suffices to prove it for elementary cycles since the height difference of each cycle is the sum of the height differences of the elementary cycles which compose it. This is done by induction on the number of cells of Λ enclosed by the cycle.

We first treat the case when the cycle follows the boundary of a hole H_i. This case is easily treated, from Lemma 1, since the figure is fully balanced (duplicated vertices create no problem). We also verify that the proposition holds for elementary cycles of length 4 around a cell.

Now, we can apply the induction argument. If we are not in the cases treated above, then the area enclosed by the cycle can be cut by a path of F, which induces two new cycles, each of them enclosing less cells of Λ than the original cycle. Thus, by induction hypothesis, the height difference of both induced cycles is null, from which it is easily deduced that the height difference of the original cycle is null. $\square$

This proposition guarantees the correctness of the definition below.

Definition 2 *Let F be a fully balanced figure , with $F = F_1 \cup F_2 \ldots \cup F_p$, where, for each integer i, F_i is a 4-connected component of F, with a fixed vertex w_i on the boundary.*

For each tiling T, the height function induced by T (denoted by h_T) is the function from the set V_F of vertices of cells of F (once necessary duplications have been done) to the set $\mathbb{Z}$ of integers, defined by $h_T(w_i) = 0$ and, for each arc (v, v') of E_F, $h_T(v') - h_T(v) = val_T(v, v')$.

Proposition 2 *For any pair (T, T') of tilings and each vertex v of F, we have : $h_T(v) - h_{T'}(v) = 0[4]$.*

Proof. obvious by induction on the length of a shortest path from w_i to v. □

Proposition 3 *Let (T, T') be a pair of tilings of F. If, for each vertex v of F, $h_T(v) = h_{T'}(v)$, then $T = T'$.*

Informally, this proposition means that a height function is a way to encode a tiling.

Proof. Let (v, v') be an edge of E_F . We have two cases:

- the line segment$[v, v']$ cuts no domino of T. Thus, $|h_T(v') - h_T(v)| = 1$,
- the line segment$[v, v']$ is the central axis of a domino of T. Thus, $|h_T(v') - h_T(v)| = 3$,

Thus, the tiling T is formed from tiles whose central axis is a segment $[v, v']$ such that $|h_T(v') - h_T(v)| = 3$. The same argument can be used for T', which yields : $T = T'$. □

The above proposition permits to define the *height distance* $\Delta(T, T')$ between two tilings by $\Delta(T, T')$ between two tilings by $\Delta(T, T') = \sum_{v \in F} |h_{T'}(v) - h_{T'}(v)|$.

1.2.5 Characterization

The proposition below gives a characterization of height functions.

Proposition 4 *Let f be a function from the set of vertices of F to the set $\mathbb{Z}$ of integers such that :*

- $f(w_i) = 0$,
- *for each arc (v, v') of E_F such that $sp(v, v') = 1$, either $f(v') - f(v) = 1$ or $f(v') - f(v) = -3$,*
- *if, moreover, the arc (v, v') is on the boundary of F, then $f(v') - f(v) = +1$.*

There exists a tiling T such that $f = h_T(v)$.

Proof. Let $(v_0, v_1, v_2, v_3, v_4 = v_0)$ be a cycle around a white cell, counterclockwise. The second constraint of the proposition implies that we have three vertices v_i such that $f(v_{i+1}) - f(v_i) = 1$ and a unique vertex v_j such that $f(v_{j+1}) - f(v_j) = -3$. One easily obtains a symmetric condition for black cells.

Thus, the set T of dominoes which are cut into both halves by an edge whose extremities, say v and v', are such $|f(v) - f(v')| = 3$, is a tiling of F. One obviously verifies (by induction on the distance from w_i to v) that, for each vertex v of F, $f(v) = h_T(v)$. □

1.2.6 Flips and height function

The vertices around which a flip can be done are easily characterized with the height function.

Definition 3 *Let T be a tiling of F. A* local maximum *(respectively* minimum*) of T is an interior vertex v of F such that, for each neighbor v' of v, $h_T(v') < h_T(v)$ (respectively $h_T(v') > h_T(v)$).*

Proposition 5 *An interior vertex v of F is a local extremum of T if and only if v is the center of a 2×2 square S which is exactly covered by two dominoes of T, with a common large side.*

Proof. If v is a local minimum, let v' and $''$ be the neighbors of v such that $sp(v', v) = sp(v'', v) = 1$. Notice that v is the middle of the line segment $[v', v'']$. Since one cannot have : $h_T(v') = h_T(v) + eq(v, v') - 1$, the equality $h_T(v') = h_T(v) + eq(v, v') + 3$ necessarily holds, thus the domino whose symmetry axis is $[v, v']$ is a domino of T. The same argument holds with v'', which yields that S is exactly covered by dominoes of T. A similar proof can be done for a local maximum of T.

Conversely, assume that S is exactly covered by dominoes of T. One easily sees, applying rules which define a height function, that v is a local extremum of T. □

What are the consequences of a flip around a vertex v on the height function ? For each vertex v' such that $v' \neq v$, we have $h_T(v') = h_{T_v}(v')$, since there exists a path of F from w_i (the origin vertex of the connected component containing v') to v' which does not pass through v.

For v, we have $|h_T(v) - h_{T_v}(v)| = 4$. If $h_{T_v}(v) = h_T(v) + 4$, then we say that the flip is going up (note that the local minimum v is transformed into a local maximum), and, if $h_{T_v}(v) = h_T(v) - 4$, then we say that the flip is going down (the local maximum v is transformed into a local minimum). Moreover, if $h_{T_v}(v) = h_T(v) + 4$, we say that h_{T_v} *covers* h_T.

2 Lattices of tilings and stability

2.1 Lattice structure

Height functions canonically induce an order on the set Γ_F of tilings of the fully balanced figure F. Given a pair (T, T') of tilings of F, we say that $T \leq T'$ if and only if, for each vertex v of F, $h_T(v) \leq h_{T'}(v)$.

Proposition 6 *Let (T, T') be a pair of tilings of F. The functions $f = inf(h_T, h_{T'})$ and $f' = sup(h_T, h_{T'})$ are height functions of tilings.*

Proof. We prove this proposition for f (the proof for f' is similar) using proposition 4. The first and last constraints are obviously satisfied, since $h_T(v') - h_T(v) = h_{T'}(v') - h_{T'}(v)$ for each arc (v, v') such that $[v, v']$ is included in the boundary of F.

Let (v, v') be an arc of E_F such that $sp(v, v') = 1$. Assume that $h_T(v) < h_{T'}(v)$. Thus, from proposition 2, we have : $h_T(v) \leq h_{T'}(v) - 4$. On the other hand, either $h_T(v') = h_T(v) + 1$ or $h_T(v') = h_T(v) - 3$. Thus :

$$h_T(v') \leq h_T(v) + 1 \leq (h_{T'}(v) - 4) + 1 = h_{T'}(v) - 3$$

Moerover, either $h_{T'}(v) = h_{T'}(v') - 1$ or $h_{T'}(v) = h_{T'}(v') + 3$, thus

$$h_{T'}(v) - 3 \leq (h_{T'}(v') + 3) - 3 = h_{T'}(v')$$

We have proven that if $h_T(v) < h_{T'}(v)$ then $h_T(v') \leq h_{T'}(v')$. Consequently, $f(v') - f(v) = h_T(v') - h_T(v)$, which guarantees the second constraint of proposition 4.

The case when $h_T(v) > h_{T'}(v)$ can be treated with the same kind of argument, and the case when $h_T(v) = h_{T'}(v)$ is obvious. □

A clear consequence of this proposition may be stated in the language of order theory (see [6] or [8]):

Corollary 1 *The order $\leq$ induces a structure of distributive lattice on the set Γ_F of tilings of F.*

For the following, for each pair (T, T') of tilings of F, the tiling whose height function is $inf(h_T, h_{T'})$ (respectively $sup(h_T, h_{T'})$) is denoted by $inf(T, T')$ (respectively $sup(T, T')$).

Definition 4 *Let (T, T') be a pair of tilings of F. We say that T and T' are* boundary equivalent *(denoted by $T = T'[\delta F]$) if, for each vertex v of the boundary of F, $h_T(v) = h_{T'}(v)$.*

Remark that if $T = T'[\delta F]$, then $T = inf(T, T')[\delta F]$ and $T = sup(T, T')[\delta F]$

Proposition 7 *Given a pair (T, T') of boundary equivalent tilings of F, we have $T \leq T'$ if and only if there exists a sequence $(T_0, T_1, \ldots, T_p)$ of tilings of F such that $T_0 = T$, $T_p = T'$ and, for each integer i such that $0 \leq i < p$, T_{i+1} covers T_i.*

Proof. The converse part of the first part of the proposition is obvious. The direct part of the proposition is proven by induction in the quantity $\Delta(T, T')$.

The result is obvious if $\Delta(T, T') = 0$. Now, assume that $\Delta(T, T') > 0$, $T \leq T'$ and $T = T'[\delta F]$. We have to prove that there exists a vertex w such that $h_T(w) < h_{T'}(w)$ (which yields that $h_T(w) \leq h_{T'}(w) - 4$ from Proposition 2), and an upward flip can be done from T around w.

It suffices to take w such that $h_{T'}(w) - h_T(w)$ has the maximal value and, moreover $h_T(w)$ is minimal with the previous condition (notice that w is not on the boundary of F since the tilings are boundary equivalent). Let w' be a neighbor of w. If $h_T(w') < h_T(w)$, then we necessarily have $h_{T'}(w) - h_T(w) \leq h_{T'}(w') - h_T(w')$, which contradicts the definition of w. Thus w is a local minimum of h_T and an upward flip can be done around w. □

Corollary 2 *Let (T, T') be a pair of tilings of F. We have $T = T'[\delta F]$ if and only if T and T' are connected by flips.*

Moreover, in such a case, we have $d(T, T') = \Delta(T, T')/4$ and a tiling T'' is on a geodesic between T and T' if and only if $h_{min(T,T')} \leq h_{T''} \leq h_{sup(T,T')}$.

Proof. The first part of the proposition is easily proved using $inf(T, T')$ and the previous propositition.

For the equality, first see that $d(T, T') \geq \Delta(T, T')/4$ since a flip decreases $\Delta(T, T')$ from at most 4 units. Afterwards, If $T \leq T'$, then any sequence of increasing flips from T to T' has length $\Delta(T, T')/4$, which proves the equality in this case. The general case follows, once it has been noticed that $\Delta(T, T') = \Delta(T, inf(T, T')) + \Delta(inf(T, T'), T')$.

Now, each tiling T'' on a geodesic between T and T' is such that $h_{min(T,T')} \leq h_{T''} \leq h_{sup(T,T')}$ since each flip on a geodesic has to decrease the heigth distance from 4 units. Conversely, take a tiling T'' such that $h_{min(T,T')} \leq h_{T''} \leq h_{sup(T,T')}$. We denote by V_1 the set of vertices v such that $h_{inf(T,T')}(v) = h_T(v)$ (which yields that $h_{sup(T,T')}(v) = h_{T'}(v)$) and V_2 the set of vertices v such that $h_{inf(T,T')}(v) = h_{T'}(v)$ (and $h_{sup(T,T')}(v) = h_T(v)$). If a vertex v is element of $V_1 \cap V_2$, then $h_T(v) = h_{T'}(v) = h_{T''}(v)$. We have:

$$\Delta(T, T'') = \sum_{v \in V_1} (h_{T''}(v) - h_T(v)) + \sum_{v \in V_2} (h_T(v) - h_{T''}(v))$$

$$\Delta(T'', T') = \sum_{v \in V_1} (h_{T'}(v) - h_{T''}(v)) + \sum_{v \in V_2} (h_{T''}(v) - h_{T'}(v))$$

Thus adding these equalities, we obtain:

$$\Delta(T, T'') + \Delta(T'', T') = \sum_{v \in V_1} (h_{T'}(v) - h_T(v)) + \sum_{v \in V_2} (h_T(v) - h_{T'}(v)) = \Delta(T, T')$$

which proves that T'' is on a geodesic between T and T'. □

Corollary 3 *For each pair (T, T') of tilings defining a space of tilings, we have $G_{(T,T')} = G_{(inf(T,T'),sup(T,T'))}$.*

Proof. It is a clear consequence of Corollary 2 for the spaces of tilings defined in Subsection 1.2.3. □

Corollary 4 *Each boundary equivalence class is the set of vertices of a space of tilings.*

Proof. Since each class is finite, it has a unique minimal tiling and a unique maximal tiling. The class is the set of vertices of the space of tilings generated by these two tilings. □

2.2 Stability of the class of lattices of tilings

Definition 5 *Let (T, T') be a pair of tilings of a same figure F such that $T \leq T'$ and T and T' are boundary equivalent. The lattice formed from tilings T of F such that $T \leq T'' \leq T'$ is denoted by $L_{(T,T')}$.*

The class of lattices L such that there exists a pair (T, T') of tilings such that $T \leq T'$ and $L = L_{(T,T')}$, is denoted by Ψ (we identify isomorphic lattices in Ψ).

We now study the properties of the class Ψ.

Proposition 8 (reversing stability) *If L is an element of Ψ, then the dual lattice of L (i. e. the lattice obtained reversing the order) is also an element of Ψ.*

Proof. It suffices to remark that a translation of $(1,0)$ reverses the order, since this translation reverses the colors of cells, and consequently, the direction of flips.

Precisely, notice that if $T_1 \leq T \leq T_2$, then $T_2 + (1,0) \leq T + (1,0) \leq T_1 + (1,0)$: tilings $L_{(T_1,T_2)}$ and $L_{(T_2+(1,0),T_1+(1,0))}$ are dual. □

Proposition 9 (product stability) *If L and L' both are elements of Ψ, then the lattice product of $L \times L'$ also is an element of Ψ.*

Proof. Let state $L = L_{(T_1,T_2)}$ and $L' = L_{(T'_1,T'_2)}$. Notice that we also have $L' = L_{(T'_1+(i,j),T'_2+(i,j))}$, for each pair (i,j) of $\mathbb{Z}^2$ such that $i+j$ is even.

Now, choose a pair (i_0,j_0) of $\mathbb{Z}^2$, with i_0+j_0 even, such that T_1 and $T'_1 + (i_0,j_0)$ do not cover a same cell. The product $L \times L'$ is isomorphic to $L_{(T_1 \cup (T'_1+(i_0,j_0)),T_2 \cup (T'_2+(i_0,j_0)))}$. □

Now we present what we call the *tilting property*. Let (T,T') be a pair of tilings connected by flips. The space of tilings $G_{(T,T')}$ can canonically be directed to obtain an order, using geodesics between T and T', as follows: let (T_1,T_2) be a pair of elements of $G_{(T,T')}$. We say that $T_1 \leq_{geod} T_2$ if $d(T,T_2) = d(T,T_1) + d(T_1,T_2)$.

Proposition 10 (tilting stability) *The order $\leq_{geod}$ on $G_{(T,T')}$ is a distributive lattice, which belongs to the class Ψ.*

Proof. We say that a cell c of F is a *negative* cell, (respectively *positive* cell) if there exists a corner v of c such that $h_T(v) < h_{T'}(v)$ (respectively $h_{T'}(v) < h_T(v)$).

Let $(v_0,v_1,v_2,v_3,v_4 = v_0)$ be a contour cycle of a cell c such that, for each integer i of $\{0,1,2,3\}$, $sp(v_i,v_{i+1}) = 1$. We recall that there exists a unique integer i_0 such that $h_T(v_{i_0+1}) = h_T(v_{i_0}) - 3$, and for $i \neq i_0$, $h_T(v_{i+1}) = h_T(v_i) + 1$.

Thus, if $h_T(v_0) > h_{T'}(v_0)$ (i. e. $h_T(v_0) \geq h_{T'}(v_0) + 4$) from Proposition 2), we also have $h_T(v_{i_0}) > h_{T'}(v_{i_0})$ and, for each integer i, $h_T(v_i) \geq h_{T'}(v_i)$. The first inequality proves that if c is a positive cell, then the domino of T (respectively T') which covers c also covers another positive cell. The second one ensures that a cell cannot be simultaneously positive and negative.

Thus we can state $F = F_+ \cup F_- \cup F_=$, where F_+ denotes the figure formed from positive cells, F_- denotes the figure formed from negative cells, and $F_=$ denotes the figure formed from remaining cells. Notice that each of these subfigures is fully balanced since it can be obtained from F removing dominoes.

We also can state $T = T_+ \cup T_- \cup T_=$ and $T' = T'_+ \cup T'_- \cup T'_=$, with corresponding notations, i. e. T_- and T'_- both are tilings of F_- such that $T_- < T'_-$, T_+ and T'_+ both are tilings of F_+ such that $T'_+ < T_+$, and $T_= = T_=$ is a tiling of $F_=$.

Moreover, for each vertex v of the boundary of F_+ (or F_-), we necessarily have $h_T(v) = h_{T'}(v)$. Thus, from Proposition 2, each tiling T'' of $G_{T,T'}$ can be partitioned as follows : $T'' = T''_+ \cup T''_- \cup T''_=$, with $T_- \leq T''_- \leq T'_-$, T_-, T'_- and T''_- are boundary equivalent, $T'_+ \leq T''_+ \leq T'_+$, T_+, T'_+ and T''_+ are boundary equivalent, and $T_= = T'_= = T''_=$.

Now, take a pair (i,j) of elements $\mathbb{Z}^2$ such that $i+j$ is odd and $F_- \cap (F_+ + (i,j))$ is the empty set. The function ϕ from $G_{T,T'}$ to $L_{(T_- \cup (T_+ + (i,j)),T'_- \cup (T"_+ + (i,j)))}$ defined by : $\phi(T) = T_- \cup (T_+ + (i,j))$ is an order isomorphism. □

3 Tilting

3.1 Stability of classes of orders

The construction that has been presented in the previous section and which consists in orienting geodesics between two elements is not specific to the spaces of tilings. It can be defined for any graph and thus raises several questions.

Let $G = (V,E)$ be an undirected graph and (s,t) be a pair of vertices of G We define the set $G_{s,t}$ formed from the vertices v of G such that there exists a sequence $(s = v_0, v_1, \ldots, v_p = t)$ (called a *geodesic*), of minimal length, such that for each integer i such that $1 \leq i < p$, (v_i, v_{i+1}) is an edge of G.

The set $G_{(s,t)}$ can be ordered as follows : let (v, v') be a pair of vertices of $G_{(s,t)}$. We say that $v \leq_{(s,t)} v'$ if there exists a geodesic, from s to t, passing through v and, afterwards, through v'.

For any vertex x of G, we also define the G_x formed from the vertices v of G such that there exists a geodesic from x to v. As previously, we can order this set G_x as follows : let (v, v') be a pair of vertices of G_x. We say that $v \leq_x v'$ if there exists a geodesic, from x to v', passing through v.

These operations on graphs (that we call *tilts*) may be extended to any order P by considering for any pair (s,t) of elements of P, the order $P_{(s,t)} = (G_{(s,t)}, \leq_{(s,t)})$ where $G = G(P)$ the undirected graph obtained from the transitive reduction of P. In the same way, for any element x of P, we define $P_x = (G_x, \leq_x)$ where $G = G(P)$.

Definition 6 *A class Φ of orders is stable if for any order P of Φ and any pair (s,t) of elements of L, the order $(P_{(s,t)}, \leq_{(s,t)})$ still belongs to Φ.*

We may wonder which order properties are preserved when applying a tilt between two elements. Some properties may be lost: for instance, as shown on Figure 3, for some lattices L there exist elements s, t such that $L_{(s,t)}$ is not a lattice.

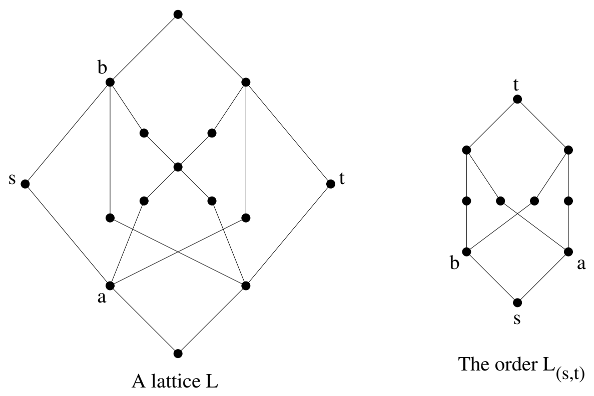


Figure 3: A lattice L and two vertices s, t such that $L_{(s,t)}$ is not a lattice.

However there exist some stable classes of lattices. In the previous section, we proved that the class of lattices of tilings is stable. Some other examples will be presented in the next subsections. Some stable classes may be defined on an underlying graph according to the following scheme.

Proposition 11 *Let $G = (V, E)$ be an undirected graph and Φ the class of all orders $(G_{(s,t)}, \leq_{(s,t)})$ where $s, t \in V$. If for all $s, t \in V$ and $u, v \in G_{(s,t)}$ we have $(G_{(s,t)})_{(u,v)} = G_{(u,v)}$, then Φ is stable.*

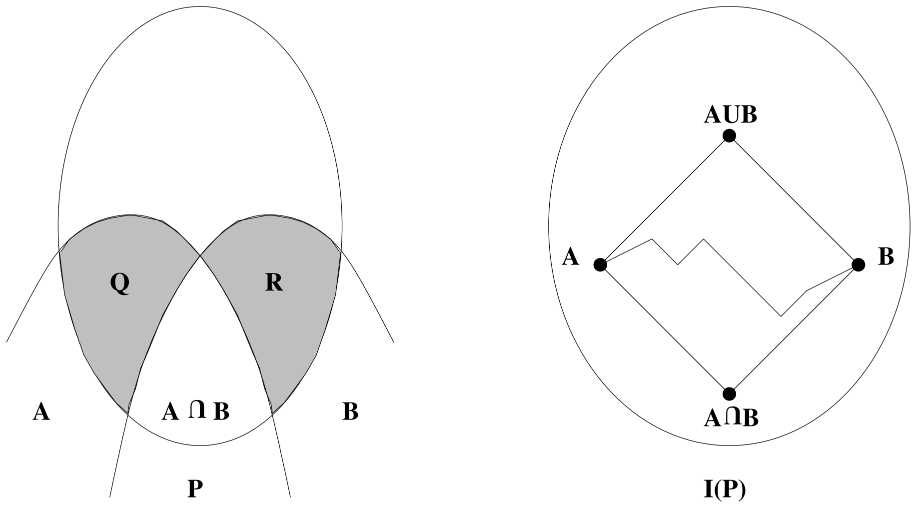
3.1.1 Distributive lattices

Proposition 12 *The class of distributive lattices is stable.*

Proof. Thanks to the Birkhoff's representation theorem mentioned in Section 1.1, the distributive lattice L is isomorphic to the lattice of ideal $I(P)$ of an order P. The graph $G(L)$ is fully described thanks to the following property: in the transitive reduction of $I(P)$ there is a directed edge from the ideal I to the ideal I' if and only if $I \subseteq I'$ and $|I' \backslash I| = 1$.

Given two elements of L, we consider them as two ideals A and B of P. Due to the definition of $G(L)$, the distance between A and B in $G(L)$ is greater or equal to $|(A \backslash B) \bigcup (B \backslash A)|$. A path from A to B in $G(L)$ consists in alternatively removing from the set A the elements of $A \backslash B$ and adding the elements of $B \backslash A$ in order to reach the set B. The sequence of removals and additions is constrained by the order P so that we keep ideals along the path. More formally, if we denote by Q the order induced by P on its subset $A \backslash B$ and by R the order induced by P on its subset $B \backslash A$, we can prove that the order $L_{(A,B)}$ is isomorphic to $I(Q^d \bigcup R)$. Figure 4 represents the configuration of these sets of P and the diagram of a path from A to B in $I(P)$.

Each element of $I(Q^d \bigcup R)$ is the disjoint union $I \bigcup J$ of an ideal I of Q^d and an ideal J of R. Let ϕ be the application which associates to each element $I \bigcup J$ of $I(Q^d \bigcup R)$ the set $(A \backslash I) \bigcup J$.



The order P with its ideals A and B

A path form A to B in G(I(P))

Figure 4: Configuration in the case of a distributive lattice $I(P)$.

First we can see that this application takes its values in the set $L_{(A,B)}$. It is sufficient to notice that if J is an ideal of R, then $J \bigcup (A \bigcap B)$ is an ideal of P. If I is an ideal of Q^d, then $(A \backslash I)$ is an ideal of P as well, and finally $(A \backslash I) \bigcup J$ is an ideal of P. The definition of $G(L)$ implies that there exists in $G(L)$ a path of length $|I|$ from A to $A \backslash I$ (by removing one by one the maximal elements when they belong to I), a path of length $|J|$ from $A \backslash I$ to $(A \backslash I) \bigcup J$ (by transfering one by one the minimal elements of J from J to the ideal), a path of length $|(A \backslash I) \backslash (A \bigcap B)|$ from $(A \backslash I) \bigcup J$ to $(A \bigcap B) \bigcup J$ and a path of length $|(B \backslash J) \backslash (A \bigcap B)|$ from $(A \bigcap B) \bigcup J$ to B. It means that there is a path of length $|(A \backslash B) \bigcup (B \backslash A)|$ from A to B which passes through $(A \backslash I) \bigcup J$. It is the minimum distance between A and B in $G(L)$, thus $(A \backslash I) \bigcup J$ belongs to $L_{(A,B)}$.

The application ϕ is clearly injective, because A and R are disjoint sets. To prove that ϕ is surjective, we are going to prove that $L_{(A,B)}$ is included into the set $\{X \in I(P) \mid A \bigcap B \subseteq X \subseteq A \bigcup B\}$ which is an interval of $I(P)$ denoted by $[A \bigcap B, A \bigcup B]$. Suppose that there is a path of minimum length from A to B that passes through X where $X \notin [A \bigcap B, A \bigcup B]$. As we have already seen, the distance between A and X in $G(L)$ is equal to $|(A \backslash X) \bigcup (X \backslash A)|$ and the distance between X and B is equal to $|(X \backslash B) \bigcup (B \backslash X)|$. It is easy to prove that if $X \notin [A \bigcap B, A \bigcup B]$ then $|(A \backslash X) \bigcup (X \backslash A)| + |(X \backslash B) \bigcup (B \backslash X)| > |(A \backslash B) \bigcup (B \backslash A)|$. It means that the length of the path going through X was not the distance between A and B. It is in contradiction with our hypothesis, thus $X \in [A \bigcap B, A \bigcup B]$. In terms of additions and suppressions of elements along the path, it means that if $X \notin [A \bigcap B, A \bigcup B]$, we have for instance an element $x \in X$ such that $x \notin A$ and $x \notin B$. In order to pass through X, we have to add x at a time and later to remove x. It lengthens the path between A and B, compared to the shortest ones.

As $L_{(A,B)}$ is included into the set $[A \bigcap B, A \bigcup B]$, and If $X \in I(P)$ and $A \bigcap B \subseteq X \subseteq A \bigcup B$, then $(A \backslash X) \bigcup (X \backslash A)$ is an ideal of $Q^d \bigcup R$ and $\phi((A \backslash X) \bigcup (X \backslash A)) = X$. As $L_{(A,B)}$ is included into the set $[A \bigcap B, A \bigcup B]$, the application ϕ is surjective and the sets $L_{(A,B)}$ and $\{X \in I(P) \mid A \bigcap B \subseteq X \subseteq A \bigcup B\}$ are equal.

Concerning the orientation of the edges for $L_{(A,B)}$, the definitions of $G(L)$ and $L_{(A,B)}$ implies that a set X precedes a set Y on a path of minimum length between A and B if and only if $X \subseteq Y$ and $Y \backslash X = \{y\}$ with $y \in B$, or $Y \subseteq X$ and $X \backslash Y = \{x\}$ with $x \in A$. By replacing X and Y with $X' = (A \backslash X) \bigcup (X \backslash A)$ and $Y' = (A \backslash Y) \bigcup (Y \backslash A)$, this is exactly the transitive reduction of the lattice of the ideals of $Q^d \bigcup R$.

We can conclude that the order $L_{(A,B)}$, whose ground set is also $[A \bigcap B, A \bigcup B]$, is isomorphic to the lattice of ideals $I(Q^d \bigcup R)$ which is a distributive lattice. □

3.2 Lattices of generalized integer partitions

We provide here another example where the tilt operation may be described in details.

Let $G = (V, E)$ be a directed acyclic graph (or multigraph). A generalized integer partition (or partition for short) on G is an integer function h defined on V such that, for each edge (v, v') of E , $h(v) \geq h(v')$. The value $h(v)$ is called the number of grains in v.

Two partitions h and h' differ from a flip if there exists a vertex v_0 such that $|h(v_0) - h'(v_0)| = 1$, and for any vertex v such that $v \neq v_0$, $h(v) = h'(v)$.

The flip relation induces an undirected infinite graph S whose set of vertices is formed from partitions of our directed acyclic graph.

Lemma 2 *For each pair (h, h') of partitions of G, there exists a path from h to h' of length $\sum_{v \in V} |h(v) - h'(v)|$.*

Proof. This is done by induction on $\sum_{v \in V} |h(v) - h'(v)|$. If $\sum_{v \in V} |h(v) - h'(v)| = 0$, then $h = h'$, which gives the initialization of the induction.

If$\sum_{v \in V} |h(v) - h'(v)| \neq 0$, then one can assume without loss of generality that there exists a vertex v of V such that $h(v) < h'(v)$. Now, let v_0 be a vertex such that $h'(v_0) - h(v_0)$ is maximal.

Either a grain can be removed in v_0 for h' (i. e. $h'(v_0)$ can be decreased from one unit to obtain a new partition), or there exists a vertex v_1 such that (v_0, v_1) is an edge of G and $h'(v_0) = h'(v_1)$. Thus from the maximality condition, we have : $h(v_0) = h(v_1)$.

Thus the same argument can be repeated in v_1, and so on to create a sequence $(v_0, v_1, \ldots, v_p)$ which necessarily ends since G is finite and acyclic. A grain can necessarily be removed in v_p for h', which guarantees the induction. □

From the previous lemma, the length of a geodesic from h to h' is $\sum_{v \in V} |h(v) - h'(v)|$, since a shorter path is impossible. Hence, the set $S_{(s,t)}$ is formed from partitions h such that for each vertex v, $min(s(v), t(v)) \leq h(v) \leq max(s(v), t(v))$.

Remark 1 *Usually, only the case when s is null and t is non negative (often t being constant) is studied [9, 12, 22]. But we will see that there is no specific difficulty to take a more general framework.*

For each pair (h, h') of partitions of $S_{(s,t)}$ and each vertex v of G, whe define $cl_{(s,t,h,h')}(v)$ as the value of the pair $(h(v), h'(v))$ which is the closest from $s(v)$.

Precisely, we have $cl_{(s,t,h,h')}(v) = s(v) sign(t(v) - s(v)) min(|h(v) - s(v)|, |h'(v) - s(v)|)$.

Lemma 3 *The function $cl_{(s,t,h,h')}$ is a partition of G.*

Proof. We have to prove that, for any edge (v, v') of E, $cl_{(s,t,h,h')}(v) \geq cl_{(s,t,h,h')}(v')$. This is done by an easy case by case analysis. The only non-trivial case is (up to symmetry) when $cl_{(s,t,h,h')}(v) = h(v)$ and $cl_{(s,t,h,h')}(v') = h'(v')$.

In this case, first assume that $t(v) \geq h'(v) \geq h(v) \geq s(v)$. If $t(v') \geq h(v') \geq h'(v') \geq s(v')$, then $h(v) \geq h(v') \geq h'(v')$. Otherwise we have : $s(v') \geq h'(v') \geq h(v') \geq t(v)$, which gives $h(v) \geq s(v) \geq s(v') \geq h'(v')$.

Now, we study the opposite case when $s(v) \geq h(v) \geq h'(v) \geq t(v)$. If $s(v') \geq h'(v') \geq h(v') \geq t(v')$, then $h(v) \geq h'(v) \geq h'(v')$. If $t(v') \geq h(v') \geq h'(v') \geq s(v')$, then $h(v) \geq t(v) \geq t(v') \geq h'(v')$.

In any case, we have $h(v) \geq h'(v')$, which is the result. □

Proposition 13 *For each pair (s, t) of partitions of G, the order $(S_{(s,t)}, \leq_{(s,t)})$ has a structure of distributive lattice and the class of such lattices is stable.*

Proof. From the previous lemma, for the order $\leq_{(s,t)}$, the infimum of any pair (h, h') of partitions of $S_{(s,t)}$ exists and is equal to $cl_{(s,t,h,h')}$, and the supremum of (h, h') is $cl_{(t,s,h,h')}$.

The distributivity is a trivial consequence of the formulas below :

$$cl_{(s,t,h,h')}(v) = s(v) sign(t(v) - s(v)) min(|h(v) - s(v)|, |h'(v) - s(v)|)$$

$$cl_{(t,s,h,h')}(v) = s(v) sign(t(v) - s(v)) max(|h(v) - s(v)|, |h'(v) - s(v)|)$$

The stability is obvious, from lemma 2. □

Note that the stability has been shown for a class corresponding to the scheme of Proposition 11. In this example we can directly describe $(S_{(s,t)}, \leq_{(s,t)})$. However in order to prove stability we could also have tried to orientate the whole graph S such that all intervals are distributive lattices and then for any $s, t \in S$ we would have searched an interval containing s and t. Then the proof of Proposition 13 would have come from Proposition 12. This kind of idea will be developed in the next subsection.

3.3 Properties of the tilting operation

We have seen in Proposition 12 that for any distributive lattice L and any pair (s,t) of elements of L the order $(L_{(s,t)}, \leq_{(s,t)})$ is a distributive lattice. This condition of local distributivity is important and enables us to state a kind of reciprocal proposition.

Proposition 14 *Let G be a graph such that for any pair of vertices x,y, the order $G_{(x,y)}$ is a distributive lattice. Then for any vertex z of G, the order G_z is a meet semilattice.*

Proof.

Let $z \in G$, then for all vertices x,y of G, Claim 1 holds as a direct consequence of the definition of the orders along geodesics.

Claim 1. Let $u,v \in G_{(z,x)} \cap G_{(z,y)}$. The infimum of u and v is the same in $G_{(z,x)}$ and in $G_{(z,y)}$. Moreover $G_{(z,x)} \cap G_{(z,y)}$ is a lower half lattice.

Claim 2. Let $u \wedge v$ be the infimum of u and v of Claim 1. Then there exists a geodesic in G from u to v passing through $u \wedge v$.

To prove this fact, consider a geodesic between u and v in G. Suppose that there exists three consecutive vertices (a,b,c) on this geodesic such that a and c are closer to z than b in G (in terms of distance in G). Consider $G_{(z,b)}$, this is a distributive lattice, and thus it has the "losange" property meaning that for any $r,s \in G_{(z,b)}$, if there exists t such that $r \prec t$ and $s \prec t$, then $t = r \vee s$ and $s \wedge t \prec s$, $s \wedge t \prec t$ (see for instance [8]). In our case, we have $a \prec b$ and $c \prec b$, which implies that there exists $b' = a \wedge_{G_{(z,b)}} c$ covered by a and c. And we can transform the geodesic by replacing b by b'. By iterating this process, we get a geodesic from u to v such that at first the distance between its vertices and z decreases up to a vertex from where the distance with z increases until the geodesic reaches v. This vertex is clearly the infimum $u \wedge v$. This construction by local changes of the geodesic is illustrated on Figure 5.

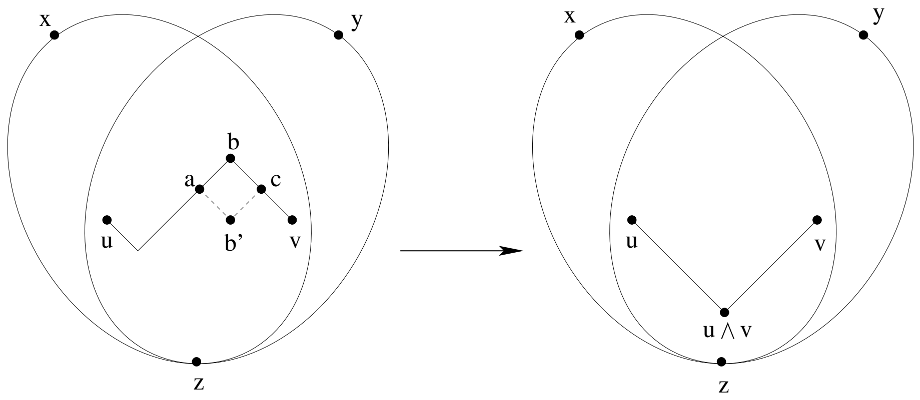


Figure 5: Constructing a geodesic from u to v passing through $u \wedge v$.

A careful look at this process shows that the constructed geodesic satisfies Claim 3.

Claim 3. All the vertices of the geodesic of Claim 2 belong to $G_{(z,x)} \cap G_{(z,y)}$.

Claim 4. If a geodesic from u to v is entirely in $G_{(z,x)} \cap G_{(z,y)}$ and it contains a sequence of vertices (a,b,c) such that b is closer to z than a and c (in terms of distance in G), then there exists a unique vertex b' adjacent to a and c in G but more distant to z than a and c. Moreover b' belongs to $G_{(z,x)} \cap G_{(z,y)}$.

As a and c belong to $G_{(z,x)}$, there exists $b' = a \vee c$ in $G_{(z,x)}$ covering a and c by the "losange" property. It implies that b' is adjacent to a and c in G and more distant to z than a and c. Now suppose that there exists a distinct b'' with the same property. Then $G_{(a,c)}$ would contain a sublattice isomorphic to M_3 (as shown on Figure 6), but that is impossible since $G_{(a,c)}$ is a distributive lattice (see for instance [8] about forbidden structures). It proves the unicity of b' satisfying the properties of Claim 4.

In the same way, as a and c belong to $T_{(z,y)}$, there exists $b'' = a \vee c$ in $T_{(z,y)}$ covering a and c. As we have just proved the unicity of such a vertex, we have $b'' = b'$ and $b' \in T_{(z,x)} \cap T_{(z,y)}$.

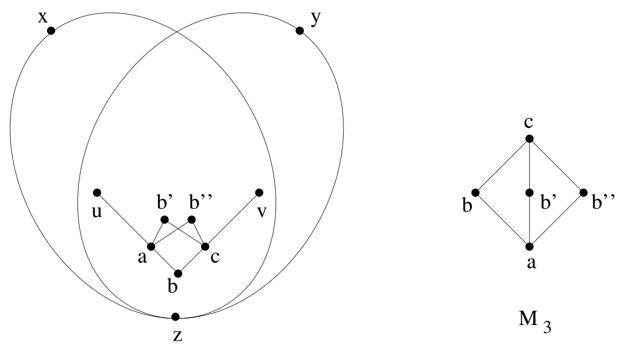


Figure 6: Forbidden configuration due to local distributivity.

Claim 5. There exists a geodesic from u to v included in $T_{(z,x)} \cap T_{(z,y)}$ and composed of two successive parts: the first part starting from u is moving away from z, the second part ending at v is moving closer to z (in terms of distance in G).

In order to prove this result, start from the geodesic constructed in Claim 2 and transform it as in Claim 2 but this time using Claim 4 to move the geodesic away from z.

Claim 6. The couple (u, v) admits a supremum $u \vee v$ belonging to $T_{(z,x)} \cap T_{(z,y)}$.

It is a consequence of the fact that $T_{(z,x)} \cap T_{(z,y)}$ is a meet semilattice and that the couple (u, v) admits an upper bound (the vertex between the two parts of the geodesic in Claim 5).

Thus we have proved that $T_{(z,x)} \cap T_{(z,y)}$ is a lattice. It has a maximum which is clearly the infimum $x \wedge y$ in G_z. $\square$

Remark 2 *The definitions of the orders G_x and $G_{(x,y)}$, as well as the results concerning semilattice and distributive lattice structures, have similarities with the studies of median graphs and median semilattices which are exposed for instance in [1, 2, 3]. However these studies hinge on the definition of median semilattices: meet semilattices where all intervals are distributive lattices and any three elements have an upper bound whenever each pair of them does. This second property makes the specificity of these studies and in our case we do not impose this condition.*

Proposition 5 gives a new insight into some known results concerning the structure of some spaces of tilings. In [19], the general case of tilings with bars of fixed length (generalizing the case of dominos which are bars of length 2) is studied. For a definition of flips similar to the one for dominos, a graph structure is induced on the set of tilings of a polygon. Concerning the graph G associated to the the set of tilings of a given polygon, it is proved that:

- for any tiling T of the polygon, the order G_T defined thanks to geodesics as previously is a meet semilattice.
- for any tilings T and T', the order $G_{(T,T')}$ defined thanks to geodesics is a distributive lattice.

Proposition 5 draws a new link between these two statements. The first one is a direct consequence of the second one, and the proposition provides arguments which are independent of the manipulated objects, namely tilings. The implication only relies on the structure of the graph G.

References

[1] S.P. Avann, *Metric ternary distributive semi-lattices*, Proc. Amer. Math. Soc. 12 (1961), p. 407-414.

[2] H.J. Bandelt, *Discrete ordered sets whose covering graphs are median*, Proc. Amer. Math. Soc. 91 (1984), n^o 1, p. 6-8.

[3] H.J. Bandelt, J.P. Barthelemy, *Medians in median graphs*, Discrete Applied Math. 8 (1984), p. 131-142.

[4] H. van Beijeren, *Exactly solvable model for the roughening transition of a crystal surface*. Phys. Rev. Lett. **38** (1977) 993–996.

[5] R. Berger, *The indecidability of the domino problem*, Memoirs of the American Mathematical Society 66 (1966).

[6] G. Birkhoff, *Lattice theory*, American Mathematical Society eds (1967).

[7] J. H. Conway, J. C. Lagarias, *Tiling with polyominoes and combinatorial group theory*, Journal of Combinatorial Theory, A 53, (1990) p. 183-208.

[8] P. A. Davey, H. A. Priestley, *An introduction to lattices and orders*, Cambridge Universitate Press (1990).

[9] N. Destainville, R. Mossery, F. Bailly, *Configurational entropy of codimension-one tilings and directed membranes* Journal of Statistical Physics, 87 (1997)

[10] C. Kenyon, R. Kenyon, *Tiling a polygon with rectangles*, proceedings of the 33^{rd} symposium on Foundations of Computer Science (1992) p. 610-619.

[11] C. Kenyon, E. Rémila, *Perfect matchings in the triangular lattice*, Discrete Mathematics (1996) vol 152, p. 191-210.

[12] M. Latapy, *General integer partitions, tilings of zonotopes and lattices*, proceedings of the conference Formal Power Series and Algebraic Combinatorics (FPSAC) (2000) Springer 256-267.

[13] C. Moore, I. Rapaport, E. Rémila, *Tiling groups for Wang tiles*, proceedings of the 13^{th} annual ACM-SIAM Symposium On Discrete Algorithms (SODA) SIAM eds, (2002), to appear.

[14] J. G. Propp, *Lattice structure for orientations of graphs*, preprint.

[15] J.G. Propp, *The many faces of alternating-sign matrices*, procceedings of the 1^{st} international conference on Discrete Models, Combinatorics, Computation and Geometry, in Discrete Mathematics and Theoretical Computer Science.

[16] E. Rémila, *Tiling groups : new applications in the triangular lattice*, Discrete and Computational Geometry (1998) 20, p. 189-204.

[17] E. Rémila, *An algebraic method to compute a shortest path of local flips between two tilings*, proceedings of the 11^{th} annual ACM-SIAM Symposium On Discrete Algorithms (SODA) SIAM eds, (2000) p. 646-653.

[18] E. Rémila, *On the lattice structure of the set of tilings of a simply connected figure with dominoes*, proceedings of the 3^{rd} International Conference on Orders, Algorithms and Applications (ORDAL) (1999) LIP Research Rapport 1999-25, ENS Lyon.

[19] E. Rémila, *On the structure of some spaces of tilings*, SIAM J. Discrete Math. (2002) vol. 16, n^o 1, p. 1-19.

[20] E. Rémila, *Lattices of tilings : an extension to figures with holes*, (2001) LIP Research Rapport 2001-48.

[21] N. C. Saldanha, C. Tomei, M. A. Casarin Jr, D. Romualdo, *Spaces of domino tilings*, Discrete Computational Geometry (1995) vol 14, p. 207-233.

[22] R. P. Stanley, *Ordered structures and partitions*, Memoirs of the ACM, 119 (1972)

[23] N. Thiant *A linear algorithm for the domino tiling of a plane picture whith abounded numbers of holes* to appear in Theoretical Computer Science.

[24] W. P. Thurston, *Conway's tiling group*, American Mathematical Monthly, (1990) p. 757-773.