Computation of balanced separations

1 Balanced separation

Input: a partially ordered set T with n vertices.
Output: two subsets S_1 and S_2 of T such that there does not exist vertices $x_1 \in S_1$ and $x_2 \in S_2$ where $x_1 \leq_T x_2$ or $x_2 \leq_T x_1$ (in particular it implies that $S_1 \cap S_2 = \emptyset$).
Optimization: maximize $\min(|S_1|, |S_2|) = f(|S_1|, |S_2|)$.

2 The case of trees: polynomial

▷ **Dynamic programming:** let A be a subtree of the initial tree T, associate with the root of A the set $\mathcal{S}(A)$ of all solutions $(|S_1|, |S_2|)$ corresponding to the acceptable ouputs (S_1, S_2) for A (and not only solutions maximizing f).

Since $0 \leq |S_1| \leq n$ and $0 \leq |S_2| \leq n$, it is clear that $|\mathcal{S}(A)| \leq n^2$. This polynomial bound is sufficient to achieve dynamic programming in polynomial time. However it is possible to reduce the set of couples stored at the root of A: by choosing an appropriate order $\prec$ on $\mathcal{S}(A)$ such that f is increasing ($(p, q) \prec (r, s) \implies f(p, q) \leq f(r, s)$), it is sufficient to store the solutions which are maximal for $\prec$.

Without loss of generality, we can permute S_1 and S_2 and thus each couple $(|S_1|, |S_2|)$ is seen as an unordered pair. Let $p \leq q$ and $r \leq s$, we define $\prec$ by $(p, q) \prec (r, s)$ if $p \leq r$ and $q \leq s$. Our optimization function f is clearly increasing with respect to $\prec$. We denote by $\mathcal{S}^*(A)$ the solutions in $\mathcal{S}(A)$ maximal for $\prec$, we have $\mathcal{S}^*(A) \leq n$

▷ **Construction of the maximal solutions $\mathcal{S}^*(A)$ for any subtree A**

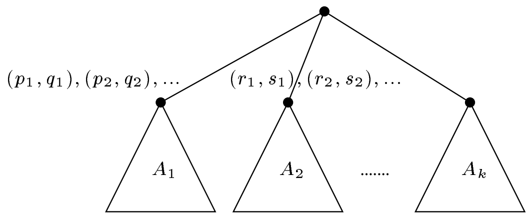

Figure 1: Construction of solutions for A from solutions for its subtrees A_1,...,A_k

Let A_1, A_2, ..., A_k be the non-empty subtrees of A (corresponding to the children of its root as shown on Figure 1).

- Merge the solutions of $\mathcal{S}^*(A_1)$ and $\mathcal{S}^*(A_2)$ by generating for each $(p_i, q_i) \in \mathcal{S}^*(A_1)$ and $(r_j, s_j) \in \mathcal{S}^*(A_2)$ the two solutions $(p_i + r_j, q_i + s_j)$ and $(p_i + s_j, q_i + r_j)$ (corresponding to the two possible unions of the solutions for A_1 and A_2). Thus we have constructed $\leq n^2$ solutions from which we then remove the solutions which are not maximal for $\prec$. Overall complexity of this step: $O(n^2)$ time and space.

- Merge this pruned set of solutions with $\mathcal{S}^*(A_3)$ using the same process, and so on with the next sets of solutions until $\mathcal{S}^*(A_k)$ is merged.

- Add to the final set of solutions the couple $(0, |A|)$ (and remove the solutions that it may cover), return this set as $\mathcal{S}^*(A)$.

$\triangleright$ **The whole algorithm for the initial tree T**

1. Initialize by associating with each leaf x the couple $(0, 1)$ (corresponding to $S_1 = \emptyset$ and $S_2 = \{x\}$).

2. Bottom-top algorithm from the leaves to the root of T computing $\mathcal{S}^*(A)$ for any subtree A as above.

3. End by choosing the best solution in $\mathcal{S}^*(T)$.

The complexity of this algorithm is $O(n^3)$ time and space ($O(n^2)$ per vertex).

Remark 1. *One couple of subsets (S_1, S_2) may be generated and stored for each couple $(|S_1|, |S_2|)$ associated with any vertex. It does not change the complexity (up to a constant factor). Risks of exponential size if we want to store all the subsets (S_1, S_2) corresponding to each couple of integers (but might be avoided).*

Remark 2. *Is it the best complexity ?*

Remark 3. *The space $[0, n] \times [0, n]$ where the couples $(|S_1|, |S_2|)$ take their values is fundamental to bound the number of partial solutions. The natural extension of the problem with weights on vertices is NP-complete as a generalization of* 2-PARTITION.

3 The general case: *NP*-complete

Theorem 4. *The decision problem associated with* BALANCED SEPARATION *is NP-complete for the class of bipartite partially ordered sets.*

Sketch of proof. The following reduction starts from an instance of BALANCED COMPLETE BIPARTITE SUBGRAPH (GT24 in Garey & Johnson), namely a bipartite graph $B = (X, Y, E)$ and an integer $K > 0$ (without loss of generality, we suppose $|X| = |Y| = N$). This problem consists in looking for $U \subseteq X$ and $V \subseteq Y$ such that $|U| \geq K$, $|V| \geq K$ and $u \in U$, $v \in V$ implies $\{u, v\} \in E$.

Consider $\bar{B} = (X, Y, \bar{E} = X \times Y - E)$ the complementary of B. Add $2N$ vertices X' to X and $2N$ vertices Y' to Y, and build the bipartite graph (that can be seen as a partially ordered set) $B' = (X \cup X', Y \cup Y', \bar{E} \cup (X' \times Y) \cup (X \times Y'))$ as shown on Figure 2. We can prove that there exists a balanced complete bipartite subgraph of B of size $\geq K$ if and only if there exists a balanced separation (S_1, S_2) of B' such that $\min(|S_1|, |S_2|) \geq 2N + K$.

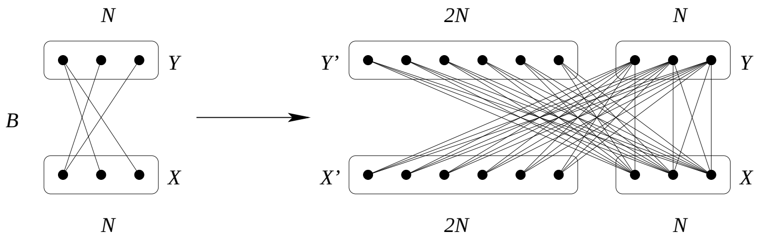

Figure 2: Construction of B' from B, which is polynomial

Let (U, V) be a complete bipartite subgraph of B with $U \subseteq X$, $V \subseteq Y$, $|U| \geq K$, $|V| \geq K$. Then $(U \cup Y', V \cup X')$ is a balanced separation of B' where each subset is $\geq 2N + K$.

Now let (S_1, S_2) be a balanced separation of B' such that $|S_1| \geq 2N + K$, $|S_2| \geq 2N + K$. If $S_1 \cap X' \neq \emptyset$ (resp. $S_1 \cap Y' \neq \emptyset$), then $S_2 \cap Y = \emptyset$ (resp. $S_2 \cap X = \emptyset$). The same holds when inverting the roles of S_1 and S_2. Since $|S_1| > 2N$, we have $S_1 \cap (X' \cup Y') \neq \emptyset$. For the same reason, $S_2 \cap (X' \cup Y') \neq \emptyset$. However we can't have simultaneously $S_1 \cap X' \neq \emptyset$, $S_1 \cap Y' \neq \emptyset$, $S_2 \cap X' \neq \emptyset$ and $S_2 \cap Y' \neq \emptyset$, otherwise $S_1 \subseteq X' \cup Y'$, $S_2 \subseteq X' \cup Y'$ and $|S_1| + |S_2| \leq 4N$ which is not true. Thus, without loss of generality, we can suppose that $S_1 \cap X' = \emptyset$, it implies $S_1 \cap Y' \neq \emptyset$ and $S_2 \cap X = \emptyset$. Since $|S_1 \cup S_2| > 4N$, we also have $(S_1 \cup S_2) \cap X' \neq \emptyset$, thus necessarily $S_2 \cap X' \neq \emptyset$ and $S_1 \cap Y = \emptyset$. Consequently $S_1 \subseteq Y' \cap X$, and since $|S_1| > 2N$, $S_1 \cap X \neq \emptyset$ which implies $S_2 \cap Y' = \emptyset$. Finally $S_1 \subseteq Y' \cup X$ and $S_2 \subseteq X' \cup Y$, and it is easy to check $(S_1 \cap X, S_2 \cap Y)$ corresponds to a complete bipartite subgraph of B of size $\geq K$. $\square$

Remark 5. *The* BALANCED SEPARATION *optimization problem for partially ordered sets may be generalized to undirected graphs as follows: given an undirected graph $G = (V, E)$, find a subset $S \subseteq V$ partitioning $G - S$ into two non-adjacent subsets S_1 and S_2 maximizing* $\min(|S_1|, |S_2|)$. *This is exactly our initial problem by considering the comparability graph of the partially ordered set. The subset S is called a separator, approximation algorithms for computing other kinds of separators could be useful ...*