Extending a poset: links with its lattice of ideals

P. Baldy, M. Morvan (LIAFA, Paris), E. Thierry (LIRMM, Montpellier)

Definition : Let $(P, \leq_P)$ be a **partial order** (or **poset**) on the ground set P. A **extension** $(Q, \leq_Q)$ of $(P, \leq_P)$ is a partial order on the same ground set, $Q = P$, such that $\forall x, y \in P, x \leq_P y \Rightarrow x \leq_Q y$.

Definition : A **lattice** L is a poset such that every pair of elements has an infimum and a supremum.
A **sublattice** of L is a suborder of L which is a lattice and such that the infimum and the supremum of two elements are the same as the ones in L.

Definition : An **ideal** A of the poset $(P, \leq_P)$ is a subset of P such that $\forall x, y \in P, (y \in A \text{ and } x \leq_P y) \Rightarrow x \in A$. The set of all the ideals of $(P, \leq_P)$ ordered by inclusion is a lattice, denoted by $I(P)$.

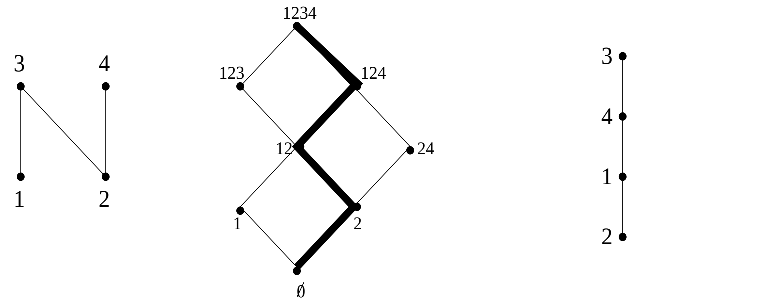
Definition : A **totally ordered set** $(P, \leq_P)$ is a poset such that $\forall x, y \in P, x \leq_P y$ or $y \leq_P x$. A **chain** of a poset $(P, \leq_P)$ is a subset of P which is totally ordered by $\leq_P$.

Definition : A **linear extension** of a poset $(P, \leq_P)$ is an extension of $(P, \leq_P)$ which is totally ordered.

Definition : A sublattice S of a lattice L is called a **chain-maximal** sublattice if every maximal chain (with respect to inclusion) of S is a maximal chain of L.

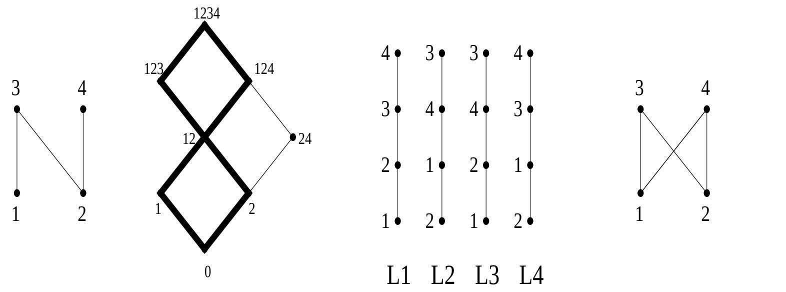
The correspondences

Theorem [Bonnet,Pouzet 1969] : Let $(P, \leq_P)$ be a partial order, the set of linear extensions of P is in one-to-one correspondence with the set of maximal chains of $I(P)$. More precisely, if L is a linear extension of P, then $I(L)$ is a maximal chain of $I(P)$. On the other hand, for all maximal chain C of $I(P)$, there exists a unique linear extension L of P such that $C = I(L)$.



A poset P — Its lattice of ideals I(P) (a maximal chain in bold) — The linear extension corresponding to this chain

Theorem [Generalization] : Let $(P, \leq_P)$ be a partial order, the set of extensions of P is in a one-to-one correspondence with the set of chain-maximal sublattices of $I(P)$.



A poset P — The lattice I(P) and a chain−maximal sublattice (in bold) — The linear extensions corresponding to the maximal chain of this sublattice — The intersection of these linear extensions

Extensions defined by forbidden structures

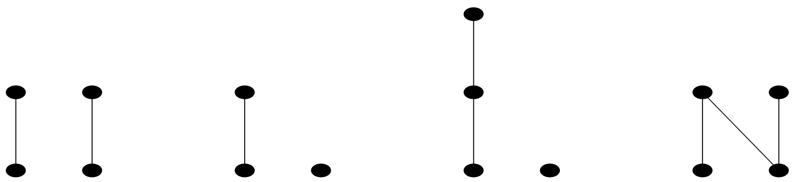
Definition : Let $(P, \leq_P)$ be a poset. A subset S of P is a **convex subset** if :

$$\forall x, y \in S, \forall z \in P, x \leq_P z \leq_P y \Rightarrow z \in S.$$

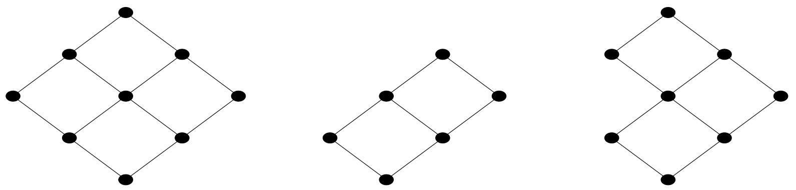
Definition : Let $\mathcal{S}$ be a set of posets. A poset is called **$\mathcal{S}$-convex-free** is it does not contain any convex subset isomorphic to a poset $(S, \leq_S)$ of $\mathcal{S}$.

Theorem : Let $(P, \leq_P)$ be a partial order and $\mathcal{S}$ a set of posets. The $\mathcal{S}$-convex-free extensions of P are in bijection with the $I(\mathcal{S})$-convex-free chain-maximal sublattices of $I(P)$, where $I(\mathcal{S})$ is the set of lattices $\{I(S) | S \in \mathcal{S}\}$.

Example : for instance, **weak order extensions** are $\mathcal{S}$-convex-free extensions with $\mathcal{S} = \{2 \dot\cup 2\}$. The figures below show some usual forbidden suborders and the associated forbidden sublattices.



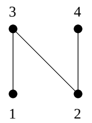
$2 \dot\cup 2$ $2 \dot\cup 1$ $3 \dot\cup 1$ **N**



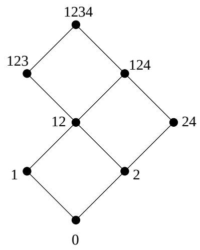
$I(2 \dot\cup 2)$ $I(2 \dot\cup 1)$ $I(N)$

Remark : The theorem of Bonnet and Pouzet can be seen as a consequence of the general theorem and the theorem about forbidden structures by taking $\mathcal{S} = \{1 \dot\cup 1\}$.

The full correspondence for an example



A poset P



Its lattice of ideals I(P)