# New Trends in Interacting Particle Systems

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**Abstract:** Since about a decade some new trends in the study of interacting particle systems can be identified. New is meant here with respect to some standard references and plays both in new mathematical challenges and in new models or phenomena. Examples are collected in the present volume. I give them here a short introduction.

## 1. Standard set-up

Recent years have seen the appearance of new types and models of interacting particle systems (IPS). New IPS do not quite fit into the mathematical framework of the standard theory or ask new types of questions or want to model phenomena investigated in more recently formulated theories. The papers included in the present volume will better explain what I mean. Fortunately, the standard references, including [4, 2], in the theory of IPS remain extremely useful and in fact, they have been able to excite the search of new boundaries and leave room for extensions and new explorations.

Even though IPS are a very natural application of Markov processes to spatially extended systems, or, global Markov processes, much impetus was gained from the interpretation in terms of multicomponent random systems. Statistical mechanics was and still is a major source of inspiration. One of the motivations for studying the stochastic Ising model was to learn more about the equilibrium statistical mechanics of magnetic systems. Other sciences like biology, economy and the social sciences have also provided effective models and the ever increasing power and speed of computers, hence simulations, have contributed considerably in making IPS a popular and a valuable tool.
Quite generally one has in mind agents or particles located at the sites of a lattice. The particles can be in different states and these are updated dynamically and depending on the states of neighboring particles. The first questions concern the definition of these models on infinite lattices and the existence of invariant or stationary measures. Usually the main questions concern the nature of the stationary states,

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their basin of attraction and ergodic properties. Some of the standard features of these IPS include

- fixed architecture: the cells, particle or spin configurations are attached to the sites of usually regular lattices. That architecture is not changing in time and has no dynamical role.
- global Markov property: the dynamics defines a Markov process and the updating is specified directly via some transition rates or probabilities. The standard formalism of linear operators is applicable for deriving the general properties of the generator and its semigroup. It implies (for compact spaces) the existence of invariant measures.
- locality of the interaction: for the given architecture, the dynamics is finite range or sufficiently damped at infinity to allow the thermodynamic limit. Mathematically that gets translated in the Feller property of the dynamics: the semigroup or the transition operator (in discrete time) leaves the set of continuous functions invariant. In fact, the dynamics can be defined directly in infinite volume.
- especially for applications in statistical mechanics there is a notion of detailed balance and reversibility that connect stationary measures with equilibrium Gibbs measures. The stochastic or kinetic Ising model is the standard example. Away from equilibrium or beyond statistical mechanics, no unique principle guides the form of the transition rules.
- simplicity of the model in its ingredients and its updating rule is seen as a quality and the ambition is often to see how a rich and complex behavior can emerge from it.

The mathematical situation is then more or less as follows: Consider a finite graph $G$ with vertex set $V$. To each $i \in V$ is assigned a variable $\eta(i) \in S$ where $S$ is finite, and a finite family $(T_i^\alpha)_\alpha$ of transformations on $\Omega_V \equiv S^V$ possibly depending (e.g. through $\alpha$) on the edges of the graph $G$. There are non-negative functions $p_i(\alpha, \cdot)$ on $\Omega_V$, called transition rates that define (what will be) the generator

$$Lf(\eta) = \sum_{i\in V} \sum_{\alpha} p_i(\alpha, \eta)[f(T_i^\alpha \eta) - f(\eta)] \tag{1.1}$$

of a Markov process on $\Omega_V$. That is quite general but standard interacting particle systems will have the extra feature that $T_i^\alpha \eta$ differs from $\eta$ only in a neighborhood of $i$ (as defined by the edge set) and that the rates $p_i(\alpha, \eta)$ are only very weakly depending on $\eta(j)$ where $j$ is far away from $i$ (in number of edges that separate $i$ from $j$). For

example, with only one $\alpha$, $T_i$ could simply change the value of $\eta(i)$ and $p_i(\eta)$ could only depend on $\eta(i)$ and $\eta(j)$ with $j$ a nearest neighbor of $i$.
It is then standard practice to define the model on infinite volume mostly regular lattices to start the study of the corresponding Feller process.

## 2. Nonlocal processes

Over the last 15 years new models have appeared that have generated a lot of interesting work where the action of the $T_i^\alpha$ is not local. I am giving the example of the abelian sandpile process, to be defined in the contribution of A. Jarai, see also the references there.
For short, see e.g. [3], in the case of sandpiles the transformations $T_i^\alpha$ can be very nonlocal. In some way the range depends on the configuration. For some configurations $\eta$, all of the degrees of freedom in the finite volume can be involved.
That nonlocality represents the major problem in defining the thermodynamic limit of the process. There are ways to deal with it but initial progress has been slow. Recently there has been great developments in obtaining the thermodynamic limit also for the sandpile process and its stationary measure on the regular lattices; a review is in [5]. I refer to the contribution of A. Jarai in the present volume for further details.

There are other processes that share similar nonlocal or long range aspects with the sandpile process. They have names like the forest fire process and the Bak-Sneppen evolution model The main open questions remain the existence of thermodynamic limits and the proof of critical properties as manifested in power law statistics for various observables, cf. [1]. The contribution by R. Meester and C. Quant gives a 'critical' discussion.

## 3. Variable or complex architecture

Recent interest is also going to interacting particle systems where the set of sites, the space or architecture, is changing in the process of interaction. Here is an example due to A. Toom of such a one-dimensional model, see [6].
At every site $i$ of $\mathbb{Z}$ there is either a plus or a minus, $\eta(i) = \pm 1$. I sketch the definition of the dynamics which is here in discrete time with parallel updating, like for probabilistic cellular automata. There are two actions. The first one is standard: every minus turns into plus with probability $\beta$ independently of others. Secondly, there is annihilation.

Informally, whenever a *word* $(+1, -1)$ occurs in the infinite configuration, it disappears with probability $\alpha$ independently of all the rest. The disappearing must be taken literary: the pair of nearest neighbor sites (with their occupation) is just removed. If one would define that on a finite configuration, the length of the total configuration would each time decrease by two. To define it mathematically for the infinite one-dimensional lattice requires somewhat more effort but the idea is clear. It can be done quite easily when starting the process from a homogeneous (translation-invariant) initial measure. Toom considers the all minus state as initial measure and considers the measure $\mu_t$ at time $t$. He proves the following kind of phase transition:
(1) the fraction of pluses in $\mu_t$ does not exceed $300\beta/\alpha^2$.
(2) if $2\beta > \alpha$, then $\mu_t$ converges to the all plus state as $t \uparrow +\infty$.
Moreover, the supremum $s(\alpha, \beta)$ of the density of pluses in $\mu_t$ over all natural $t$, is not continuous as a function of $\beta$.
These results are quite new and unexpected. See for example how the first operation favors pluses (minuses are turned into pluses) while the annihilation as such looks neutral. The analogue of $s(\alpha, \beta)$ for the contact process or for percolation is continuous; in Toom's model we see what is similar to a first order phase transition.

Similar process or processes where the architecture also has a dynamics have been considered by V. Malyshev. Sometimes the motivation is found in computer science problems but also in projects related to deep questions at the frontier of physics. Of course, one can also enjoy defining interacting particle systems on random lattices or graphs. The difference is that here there is an interaction or feedback between the architecture and the particle system itself. That can be taken into scenario's of some theories of gravity: the particle configuration specifies the graph and the graph governs the interaction between and the behavior of the particles. It has even been suggested that such models with direct nonlinear feedback between architecture and dynamics are interesting toy-models in the study of quantum gravity.

The contribution of B. Derrida can partially be classified under 'special architectures' and under the next section of 'biologically inspired' IPS. There one enters the field of genomics and human evolution. Genealogy in the era of genomics also encounters models of IPS. A typical architecture is the family tree but there are random aspects to it.

## 4. Biological models

Mathematicians and physicists (and almost every kind of species in between) enjoy simplicity. The structures and rules must be simple and short to formulate; beauty is in the simplicity of the model and in the richness of possible behavior. The point is often emphasized that it is important to recognize what are the major elements in the dynamics that alone can already give rise to the required phenomena, independent of further —microscopic— details.

That attitude is not often fully appreciated by biologists. True one wants to see what is in principle possible but it does not necessarily *explain* the phenomena. Taking into account of the biological reality seems an important requirement and that reality is most of the time extremely complex and painstakingly difficult to grasp in its totality. Simple models are simply not realistic and standard practitioners of IPS will be horrified by the length to which biologists want to go in only defining the system. It is already a challenge to keep track of all the rules and elements in the models.
It is interesting to see that one can reproduce certain biological patterns by a cellular automaton or by an IPS; the pattern of a sea shell or the stripes of a zebra are typical examples. It is however not automatically true that an important insight is gained by these reproductions. A new trend in biology-related IPS is to recognize that and to go to meet the true questions of the biology community.

Biological IPS fall in the category of standard IPS except for the unusual complexity of the models itself. Examples can be tasted in the contribution of P. Hogeweg. One can ask similar questions as in IPS, e.g. for the so called kinetic or hydrodynamic limits as we see in the contribution of L. Triolo. Indeed, also biological systems have various scales of description and also there, the various levels though hierarchically connected, have a certain autonomy with possibly new and unexpected behavior. The recent trend in biophysics reaches from population dynamics and ecology to molecular and cellular biology. It is however often far form obvious to recognize what are the essential macroscopic variables and what are the relevant time-scales.
There have always been models in IPS that have been directly influenced by biology or by biological issues but the wealth of new data and new experimental techniques in biology certainly presents new inspiration for perhaps more realistic IPS.

## 5. Games and economies

The contribution of J. Miekisz gives us an introduction to spatially extended games. Here IPS inspire the definition and the study of games between multiple agents. That appears natural in the study of certain economic markets. One speaks about socio-economic systems. Again the definitions are mathematically entirely compatible with standard IPS but new concepts and new questions appear. One notion that is of particular importance is that of Nash equilibria and its stochastic stability.

Agents have at their disposal certain strategies and their payoffs in a game depend on strategies chosen both by them and by their opponents. The agents sit on the vertices of a graph and their opponents are their nearest neighbors. A Nash equilibrium is an assignment of strategies to players (thus, a configuration) such that no player, each time for fixed strategies of his opponents, could win by changing his strategy. Clearly, one feels the connection with the idea of ground states in equilibrium statistical mechanics but there are important differences. The notion of detailed balance can play a role but by the type of questions, new stochastic rules are selected with their own simplifications and properties.

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