Extending a partially ordered set: links with its lattice of ideals

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Abstract. A well-known result of Bonnet and Pouzet [3] bijectively links the set of linear extensions of a partial order P with the set of maximal chains of its lattice of ideals $I(P)$. We extend this result by showing that there is a one-to-one correspondence between the set of all extensions of P and the set of all sublattices of $I(P)$ which are chain-maximal in the sense that every chain which is maximal (for inclusion) in the sublattice is also maximal in the lattice.

We prove that the absence of an order S as a convex suborder of P is equivalent to the absence of $I(S)$ as a convex suborder of $I(P)$. Let $\mathcal{S}$ be a set of partial orders and let us call $\mathcal{S}$-convex-free any order that does not contain any order of $\mathcal{S}$ as convex suborder. We deduce from the previous results that there is a one-to-one correspondence between the set of $\mathcal{S}$-convex-free extensions of P and the set of $I(\mathcal{S})$-convex-free chain-maximal sublattices of $I(P)$. This can be applied to some classical classes of orders (total orders and in the finite case, weak orders, interval orders, N-free orders). In the particular case of total orders this gives as a corollary the result of Bonnet and Pouzet.

Key words: Partial orders, extensions, lattice of ideals, convex suborder

1. Introduction

Extending a partial order into others has been widely studied in the last decades. The following works deal with some restricted extensions classes: linear extensions (Bonnet and Pouzet [3]), minimal interval extensions (Habib et al. [5]), weak order extensions (Bertet et al. [1]), lattice extensions (MacNeille [6]), series-parallel extensions (Möhring and Radermacher [8]), or some special cases of interval extensions — and reductions — (Felsner et al. [4]).

In this paper we focus on extensions obtained by adding new comparabilities to the original order, without changing the set of vertices. The structure of this set of extensions has been studied from different points of view in [9] and [10]. A classical result in order theory states that the set of all linear extensions of a partial order is in a one-to-one correspondance with the maximal chains of the lattice of ideals of the order. Presented by Bonnet and Pouzet in [3], this result was also discovered by Stanley in [11] and [12]. A careful reading of [3] shows that the main result that will be proved in the present paper (the characterization of all extensions of an order) is a natural generalisation of the theorem stated there.

After giving definitions and notation in Section 2, we characterize in Section 3 the set of extensions of an order in terms of a one-to-one correspondence with a set of some special sublattices of its lattice of ideals. In Section 4 we characterize in the same way the set of extensions defined by special forbidden substructures. This result induces the characterization of classical classes of extensions, and in particular this gives as a corollary the characterization of linear extensions in terms of maximal chains of the lattice of ideals.

2. Definitions and notation

Let $(P, \leq_P)$ be a partial order (order for short) on the ground set P. We denote by $|P|$ the cardinal of P if it is finite. The same order relation $\leq_P$ restricted to a subset of P is called a *suborder* of P. An order on the same ground set $(Q, \leq_Q)$, $Q = P$, is called an *extension* of P if for all $x, y \in P$, $x \leq_P y$ implies $x \leq_Q y$. A *lattice* $(T, \leq_T)$ is an order such that for all $x, y \in T$, the pair $\{x, y\}$ has an infimum $x \wedge_T y$ and a supremum $x \vee_T y$. If there is no ambiguity, we will just use the notations $\wedge$ and $\vee$. A *sublattice* of T is a suborder S of T which is a lattice and such that for all $x, y \in S$, $x \wedge_S y = x \wedge_T y$ and $x \vee_S y = x \vee_T y$. A subset A of P is called an *ideal* (or *downset*) of P if $x \in A$ and $y \leq_P x$ implies $y \in A$. For instance, for all $x \in P$, $\downarrow_P x = \{y \in P, y \leq_P x\}$ is an ideal of P, which is called the *ideal generated by x in P*. We denote by $I(P)$ the set of all ideals of P ordered by inclusion. $I(P)$ is a distributive lattice, where the supremum of elements is their union, the infimum their intersection. A partial order $(P, \leq_P)$ is a *totally ordered set* if, for all $x, y \in P$, $x \leq_P y$ or $y \leq_P x$. A *chain* of a partial order $(P, \leq_P)$ is a subset of P which is totally ordered by $\leq_P$. A *linear extension* of a partial order $(P, \leq_P)$ is an extension of P which is totally ordered. If P is finite, the maximum cardinality of a chain of P minus 1 is called the *height* of P. In particular, if P is finite, the height of $I(P)$ is $|P|$. To end, a sublattice T of $I(P)$, such that every maximal chain (with respect to inclusion) of T is a maximal chain of $I(P)$, will be called a *chain-maximal* sublattice.

3. Characterization of the set of extensions

In [3], R. Bonnet and M. Pouzet have linked the linear extensions of a partial order with the maximal chains of its lattice of ideals:

THEOREM 1. *Let $(P, \leq_P)$ be a partial order, the set of linear extensions of P is in one-to-one correspondence with the set of maximal chains of $I(P)$. More precisely, if L is a linear extension of P, then $I(L)$ is a maximal chain of $I(P)$. On the other hand, for all maximal chain C of $I(P)$, there exists a unique linear extension L of P such that $C = I(L)$.*

Figure 1 shows the linear extension of P associated with a maximal chain of $I(P)$ (in bold).

Note that in the finite case, if the maximal chain C of $I(P)$ is $C = \{I_0 \subset I_1 \subset \ldots \subset I_n\}$, then the associated extension is $L = \{I_1 \setminus I_0 \leq_P I_2 \setminus I_1 \leq_P \ldots \leq_P I_n \setminus I_{n-1}\}$ ($|I_i \setminus I_{i-1}| = 1$ for all i, $1 \leq i \leq n$).

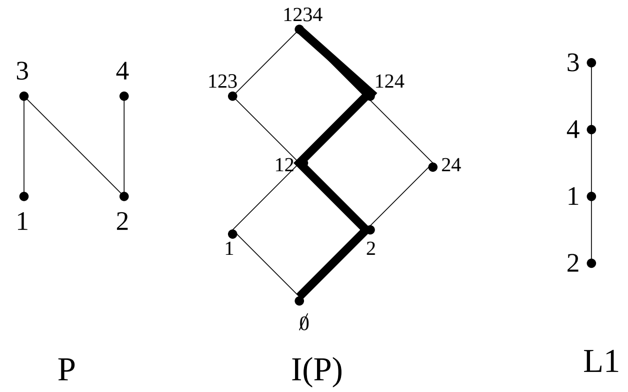

Figure 1. A partial order P, its lattice $I(P)$, a chain of $I(P)$ in bold, the corresponding linear extension of P.

This theorem can be extended to a characterization of the set of all extensions of a partial order as follows.

THEOREM 2. *Let $(P, \leq_P)$ be a partial order, the set of extensions of P is in a one-to-one correspondence with the set of chain-maximal sublattices of $I(P)$.*

Proof. We are going to prove that the application I which associates to a partial order Q its lattice of ideals $I(Q)$ draws a bijection between the extensions of P and the chain-maximal sublattices of $I(P)$.

First, the image of an extension Q of P is a chain-maximal sublattice of $I(P)$: if A is an ideal of Q, then A is an ideal of P (as Q is an extension of P), and $I(Q)$ is the sublattice of $I(P)$ because the supremum is still the union and the infimum the intersection. Now consider a maximal chain C of $I(Q)$. Thanks to Theorem 1, we can associate with C a linear extension L of Q, satisfying $C = I(L)$. Then L is an extension of Q and Q an extension of P, so L is a linear extension of P and $C = I(L)$ is a maximal chain of $I(P)$.

From Birkhoff's theorem stating the one-to-one correspondence between partial orders and distributive lattices [2], it immediately follows that I is injective. Let us prove that I is surjective. We know that each maximal chain of $I(P)$ corresponds to a linear extension of P. Let T be a chain-maximal sublattice of $I(P)$. Consider the set $\mathcal{C}$ of all the maximal chains of T. They are clearly all maximal chains of $I(P)$. To each $C \in \mathcal{C}$, we associate L_C the corresponding linear extension of P, such that $C = I(L_C)$, according to Theorem 1. Let $(Q, \leq_Q)$ be the partial order which is the intersection of all these linear extensions L_C of P, for $C \in \mathcal{C}$. Then Q is an extension of P, and we are now going to show that $I(Q) = T$, that is that the sets $I(Q)$ and T are identical since they are both ordered by inclusion.

On the first hand, let us consider x in Q. We are going to show that $\downarrow_Q x \in T$, and deduce from it that $I(Q) \subseteq T$. Let L be a linear extension of P used in the definition of Q. First, $\downarrow_L x \in T$ by the construction of L which is associated with a maximal chain of T. Then, by the definition of Q, we can say that $\downarrow_Q x$ is the intersection of the ideals $\downarrow_L x$ for all the linear extensions L used in the definition of Q. Hence $\downarrow_Q x \in T$ (because T is stable with respect to intersection). Now let A be an ideal of Q, A is the union of $\downarrow_Q x$ for all x in A. So $A \in T$ (because T is stable with respect to union). Finally $I(Q) \subseteq T$.

On the other hand, let A be an element of T. Let us show that $A \in I(Q)$. The definition of Q is equivalent to saying that $x \leq_Q y$ if and only if for all maximal chains C and for all elements B of this chain, $y \in B$ implies $x \in B$. This implies that A is an ideal of Q : we verify that $y \in A$ and $x \leq_Q y$ implies $x \in A$. So $T \subseteq I(Q)$.

For the finite case, we obtain the immediate corollary:

COROLLARY 1. *If P is finite, the set of extensions of P is in a one-to-one correspondence with the set of height $|P|$ sublattices of $I(P)$.*

Proof. If P is finite, a chain of $I(P)$ is maximal if and only if its height is $|P|$. Let T be a sublattice of $I(P)$ of height $|P|$. As T is a sublattice of $I(P)$ and since $I(P)$ is distributive, T is also distributive and all the maximal chains of T have the same height, namely $|P|$ (see [2] for this property).

Figure 2 illustrates Theorem 2: the extension Q of P is the one which is associated with the sublattice of $I(P)$ (in bold).

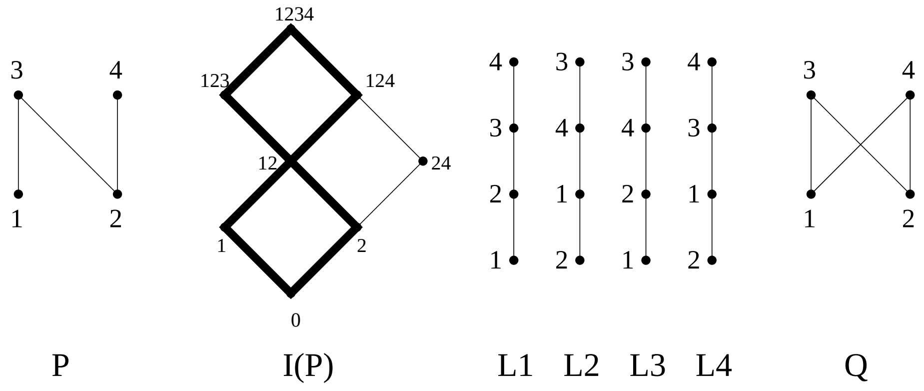

Figure 2. A partial order P, its lattice $I(P)$, a sublattice of height $|P|$ in bold, the corresponding linear extensions of P, the resulting extension Q.

4. Extensions defined by forbidden convex structures

Natural classes of partial orders are defined by forbidden substructures (linear orders, weak-orders, interval orders, series-parallel orders, N-free orders, semiorders ... see Möhring [7] for a survey). The aim of this section is to study the possibility of characterizing the set of extensions belonging to such a class in terms of chain-maximal sublattices of the lattice of ideals not containing some substructure (for example not containing the image of the initial forbidden structures). We will see that the answer is positive under additional assumptions.

Let $(P, \leq_P)$ be a partial order. A subset S of P is a *convex subset* if for all $x, y \in S$, for all $z \in P$, $x \leq z \leq y$ implies $z \in S$. The order considered on such a convex subset is the order induced by $\leq_P$. For $a, b \in P$, $[a,b]=\{x \in P, a \leq_P x \leq_P b\}$ is called an *interval* of P, which is also a convex subset of $(P, \leq_P)$. We can now state the following result:

THEOREM 3. *Let $(P, \leq_P)$ and $(S, \leq_S)$ be partial orders. P has a convex subset isomorphic to $(S, \leq_S)$ if and only if $I(P)$ has a convex subset isomorphic to $I(S)$.*

Proof. Let K be a convex subset of P isomorphic to $(S, \leq_S)$. Let $\downarrow K$ be the ideal $\{x \in P | \exists\, y \in K, x \leq_P y \text{ and } x \notin K\}$. Then the set of ideals $\{A \in I(P) | A = \downarrow K \cup \bigcup_{x \in K'} \downarrow_P x$, for $K' \subseteq K\}$ is isomorphic to $I(K)$ (the isomorphism is induced by the application $\downarrow_K x \rightarrow \downarrow_P x \cup \downarrow K$ which is injective for $x \in K$), and consequently is isomorphic to $I(S)$. It is convex in $I(P)$ thanks to the convexity of K in P.

Now let $\mathcal{K}$ be a convex subset of $I(P)$ isomorphic to $I(S)$. As $I(S)$ has a minimal and a maximal element, $\mathcal{K}$ =$[A_{min},A_{max}]$ where A_{min} and A_{max} are two ideals of $I(P)$. Consider $K = A_{max} \backslash A_{min}$. Then K is a convex subset of P: if $x, y \in K$ and $x \leq_P z \leq_P y$, then $y \in A_{max}$ implies $z \in A_{max}$, and $z \notin A_{min}$, else we would have $x \in A_{min}$, so $z \in K$. Then $\mathcal{K} =[A_{min},A_{max}]= \{A \in I(P) | A = A \cup \bigcup_{x \in K'} \downarrow_P x$, for $K' \subseteq K\}$, which is isomorphic to $I(K)$ (the isomorphism is induced by the application $\downarrow_K x \rightarrow \downarrow_P x \cup A_{min}$ which is injective for $x \in K$). So $I(S)$ and $I(K)$ are isomorphic, which implies that S and K are isomorphic.

For a given partial order, this theorem yields a bijection between the extensions with forbidden structures and sublattices of ideals with corresponding forbidden structures. Let $\mathcal{S}$ be a set of partial orders and let us call a partial order $\mathcal{S}$*-convex-free* if it does not contain any convex subset isomorphic to a partial order $(S, \leq_S)$ of $\mathcal{S}$.

COROLLARY 2. *Let* $(P, \leq_P)$ *be a partial order and* $\mathcal{S}$ *a set of partial orders. The* $\mathcal{S}$*-convex-free extensions of* P *are in bijection with the* $I(\mathcal{S})$*-convex-free chain-maximal sublattices of* $I(P)$*, where* $I(\mathcal{S})$ *is the set of lattices* $\{I(S) | S \in \mathcal{S}\}$.

In the case of finite orders, the classical classes mentionned at the beginning of this chapter are all defined by forbidden structures. However only some of them are defined by forbidden convex subsets, and hence are addressed by the previous corollary. First we give an immediate proposition concerning forbidden structures in the transitive reduction of finite orders and convex subsets (the *transitive reduction* of a finite order $(P, \leq_P)$ is the directed acyclic graph with vertex set P and where there is an edge from x to y if $x \leq_P y$ and for all $z \in P$, $x \leq_P z \leq_P y$ implies $z = x$ or $z = y$). We will see that weak-orders, interval orders and N-free orders are defined by forbidden convex subsets.

PROPOSITION 1. *Let* $\mathcal{S}$ *be a set of finite orders of height 0 or 1. A finite order whose transitive reduction does not contain any induced subgraph isomorphic to an order of* $\mathcal{S}$ *is a* $\mathcal{S}$*-convex-free order.*

We now give the definitions of weak-orders, interval orders and N-free orders, and see that this proposition relates to each of them. All the references about these classes of orders can be found in Möhring [7].

A *weak order* P is an order such that for all $x, y \in P$, if x and y are incomparable then $I(x) \setminus \{x\} = I(y) \setminus \{y\}$. The following characterization is immediate: a finite order is a weak order if and only if its transitive reduction doesn't contain any subgraph isomorphic to the graph $2 \dot{\cup} 1$ (see Figure 3).

Let P be a finite set and $(I_x)_{x \in P}$ be a set of intervals $I_x = [l_x, r_x]$ of the real line. This set defines an order on P by putting $x <_P y$ if $r_x < l_y$. An order which can be obtained in that way is called a *interval order*, and it also has a characterization by a forbidden subgraph: a finite order is an interval order if and only if its transitive reduction does not contain any subgraph isomorphic to the graph $2 \dot{\cup} 2$ (see Figure 3).

We now introduce the class of N-free orders which has various applications and several characterizations. Here we just give the definition by a forbidden subgraph. A finite *N-free order* is an order whose transitive reduction doesn't contain any subgraph isomorphic to the graph N (see Figure 3).

Proposition 1 clearly applies to these three classes of orders, and consequently Corollary 2 gives the next proposition concerning the extensions which belong to these classes.

PROPOSITION 2. *Let P be a finite order, its weak order extensions (resp. interval extensions, N-free extensions) are in one-to-one correspondence with the chain-maximal sublattices of $I(P)$ not containing any convex subset isomorphic to $I(2 \dot{\cup} 1)$ (resp. $I(2 \dot{\cup} 2)$, $I(N)$).*

Figure 4 shows these forbidden convex subsets.

Notice that totally ordered sets can also be defined (even in the infinite case) by a forbidden convex subgraph: they do not contain any subgraph with two isolated vertices (denoted by $1 \dot{\cup} 1$). This means that the linear extensions of an order P are in one-to-one correspondence with the chain-maximal sublattices of $I(P)$ which don't contain any convex subset isomorphic to $I(1 \dot{\cup} 1)$, namely the maximal chains of $I(P)$. Thanks to Theorem 3, Theorem 1 about linear extensions can be seen as a consequence of the general Theorem 2.

Some other classes of extensions, like series-parallel extensions (orders not containing a suborder isomorphic to the graph N), lattice extensions, semi-order extensions (with the forbidden $2 \dot{\cup} 2$ and $3 \dot{\cup} 1$ subgraphs, see Figure 4) do not have such characterizations. These classes of orders can't be defined by a finite set of forbidden convex suborders, so the previous theorems don't apply to them.

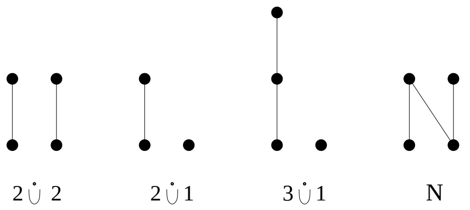

Figure 3. Some forbidden suborders.

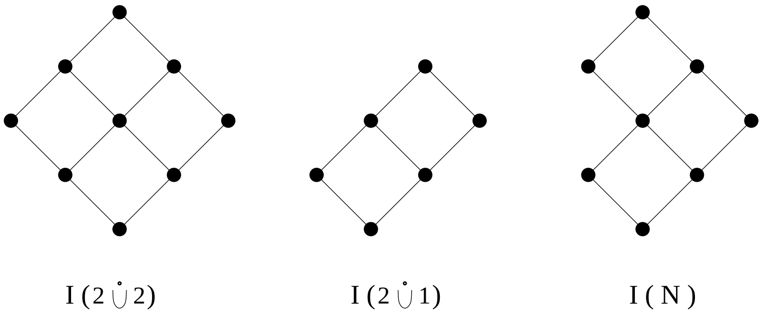

Figure 4. Some forbidden sublattices.

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