Hypergraphs and Matroids

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1 Motivation

Hypergraphs makes it possible to more compactly describe many proofs in graph theory, and may also unify several theorems in ordinary graph theory. The same holds for matroids.

Today, some futures of hypergraphs is used in computer science, notably in machine learning, and there has been a lot of research about using hypergraphs in relational databases, which might be viewed as a sort of data mining. There is also research about networks where matroids and hypergraphs are used together in the demonstrations.

I will give a short introduction to both matroids and hypergraphs in this rapport, and describe how a matroid can be represented as a hypergraph.

2 Hypergraphs

2.1 Definition and representation.

A hypergraph is a graph with edges that connects any number of nodes. If we have a set of nodes X then a hypergraph is a pair $\{X, H\}$ where $H = \{E_1, E_2, \cdots, E_m\}$ is a set of subsets of X such that $E_i \neq \emptyset$ for all i, and $\bigcup_{i=1}^{m} E_i = X$. These sets are the edges or the hyperedges of the hypergraph. The hypergraph is said simple if none of its edges is contained within another. I will most of the time note the hypergraph only by its edge-set, since the set of vertices X is often superfluous in many definitions.

2.2 Order, rank, degree, uniformity and regularity.

Some more basic definitions follows: $n(H)$ and $m(H)$ denotes the number of nodes and edges, respectively. The rank $r(H)$ of a hypergraph is defined as the maximum number of nodes in one edge, $r(H) = \max_j |E_j|$, and the anti-rank $s(H)$ is defined likewise.

We call a hypergraph uniform or r-uniform if $r(H) = s(H)$. Hence a simple graph is a 2-uniform hypergraph, and thus all simple graphs are also hypergraphs, which shouldn't come as a big surprise. The results of hypergraphs applies to simple graphs, while in the same time covering not only graphs but also other areas. However, hypergraphs has borrowed many of the concepts of usual graphs which has been transferred to hypergraphs as well.

If we define $H(x)$ as the star containing all the hyperedges that intersects x then we can define $d_H(x) = m(H(x))$ and $\Delta(H) = \max_{x \in X} d_H(x)$ follows easily. A hypergraph is said regular if $\forall x, y \in X. x \neq y, d_H(x) = d_H(y)$.

2.3 The dual of a hypergraph.

The dual hypergraph is defined as

$$H* = \{X_1, X_2, \cdots, X_n\}, X_i = \{e_j | x_i \in E_j in H\}$$

To avoid problems with this definition and possibly other definition as well, nodes not being connected to any edge is not considered as nodes, as already stated in the definition of a hypergraph. For example, if performing the conversion to a dual, a node without an edge would be considered as an edge without a node.

If we have a binary incidence matrix defined for our hypergraph, the dual can easily be obtained by taking the transpose of the matrix. $|V| = n, |E| = m$ gives a $n \times m$ matrix.

A partial hypergraph is obtained by removing a certain number of edges, and removing the nodes that no longer belongs to any hyperedge. Let $J \subset \{1, 2, \ldots, m(H)\}$ and $H = \{E_1, E_2, \ldots, E_m\}$ a hypergraph. Then the partial hypergraph H' is:

$$H' = \{E_j | j \in J\}$$

In contrast a sub-hypergraph is obtained by removing a subset of the nodes in X, which might result in the removal of edges but in general reduces their size. If $A \subset X$ where X is the node-set belonging to $H, m(H) = m$ then

$$H_A = \{E_j \cap A | 1 \leq j \leq m, E_j \cap A \neq \emptyset\}$$

Thus the partial hypergraph of H is equal to the sub-hypergraph of the dual hypergraph $H*$ and vice versa.

Only as a reminder, uniformity is concerned with the number of vertices in each edge while regularity describes the number of edges at each vertex. Since the dual exchanges the edges we have that the $\Delta(H) = r(H*)$. This leads also to that the dual of a uniform hypergraph is regular and conversely.

The equivalent to a complete graph (a clique) in hypergraphs, is shortly denoted by K_n^r, and denotes the r-uniform complete hypergraph of order n, such that $E(K_n^r) = \{E | |E| = r \wedge E \in \mathcal{P}(K_n^r)\}$.

2.4 Sperner's first theorem

Theorem 2.1 *Every simple hypergraph H of order n satisfies*

$$m(H) \leq \binom{n}{[n/2]} \tag{1}$$

$$\sum_{E \in H} \binom{n}{|E|}^{-1} \leq 1 \tag{2}$$

The proof I present here is not the proof of Sperner, but is due to Yamamoto, Meshalkin, Lubell and Bollobàs, and is presented in the book by Berge, which uses inequality (2) to simplify the proof.

Proof We start by showing inequality (2).

If we have a hypergraph (X, H) we can construct a simple directed graph G whose vertices are the subsets of X (read $V(G) \subset \mathcal{P}(X)$) and whose edges are from node A to B , $A, B \in V(G)$, if and only if $A \subset B$ and $|A| = |B| - 1$.

Thus all hyperedges is represented as a node somewhere in G, and the number of paths to a certain hyperedge $E \in H$ is $|E|!$ (Starting from $\emptyset$).

Since H is a simple hypergraph, no path passing through a hyperedge E can pass through another edge E', therefore all paths passing through node $E \in V(G)$ can not pass through any other edge's node.

Hence, the total number of paths in G from $\emptyset$ to X is

$$n! \geq \sum_{E \in H} (|E|)!(n - |E|)!$$

We get inequality (2) after some shuffling.

To solve the inequality (1) we use the fact that no binomial coefficient is greater than the central binomial coefficient, thus,

$$\binom{n}{|E|} \leq \binom{n}{\lfloor n/2 \rfloor}$$

and using inequality (2) we get

$$1 \geq \sum_{E \in H} \binom{n}{|E|}^{-1} \geq m(H) \binom{n}{\lfloor n/2 \rfloor}^{-1}$$

Further, the equalities only holds for,

1. in the even case: $n = 2h$ for the hypergraph K_n^h.
2. in the odd case: $n = 2h-1$ for either the hypergraph K_n^h or the hypergraph K_n^{h+1}.

Proof Suppose that we have a hypergraph H that satisfies inequality (1), then

$$\binom{n}{|E|} = \binom{n}{\lfloor n/2 \rfloor} \tag{3}$$

for all $E \in H$.

In the even case, $n = 2h$ all edges have the same rank, hence the hypergraph is h-uniform, and since the equality in (1) holds, we also have that $m(H) = \binom{n}{h}$ that gives us the number of edges in the graph. There is only one way to fit $\binom{n}{h}$ edges of rank h into a hypergraph, and we can conclude that $H = K_n^h$. In the odd case, $n = 2h - 1$, we have to deal with some fine-tuning to establish the proof, and I don't present it here.

2.5 Intersecting families

An intersecting family is a set of pairwise intersecting hyperedges. In a multigraph (a hypergraph with rank $r(H) \leq 2$ or, equivalent a graph with multiple edges and loops) the intersecting families are the stars and the triangles. We define $\Delta_0(H)$ as the maximum cardinality of a intersecting family in a hypergraph. Thus we get the inequality

$$\Delta_0(H) \geq \Delta(H)$$

Theorem 2.2 *Every hypergraph H of order n with no repeated edge satisfies*

$$\Delta_0(H) \leq 2^{n-1} \tag{4}$$

where the equality only holds for the hypergraph of subsets of an n-set.

2.6 Hereditary hypergraphs.

The hereditary closure $\hat{H}$ of a hypergraph (X, H) is defined as all the non-empty subsets $F \subset X$ such that $\exists E \in X.F \subset E$.

One interesting note regarding hereditary hypergraphs is Chvátal's conjecture, which still seems to be an open problem, which states that for every hereditary hypergraph we have that $\Delta_0(H) = \Delta(H)$, the cardinality of the maximal intersecting family is equal to the maximal degree of the hypergraph H. There is numerous of proofs confirming the conjecture for special cases, with additions in the latest years as well.

We will come back to hereditary hypergraphs later, but first an introduction to matroids.

3 Matroids.

The research about matroids is much older than that about hypergraphs, starting in the 1930:s with Whitney and MacLane among others. Using the definition described in Douglas B. West book *Introduction to graph theory*, a matroid is a structure obtained by restricting a hereditary system. There is many ways to define a matroid, and this way gives something of a bird's view of the different definitions.

3.1 Hereditary systems.

A hereditary family is a collection of sets such that every subset of a set in such a family is also contained in the family. A hereditary system M on a set E is a hereditary family $A_M \subset \mathcal{P}(E)$ such that A_M is nonempty, and such that A_M, to be called an aspect of M, is defined in one of the following ways:

- I_M, the independent sets of M.
- C_M, the circuits of M, that is to say the minimal sets in D_M, being the set of all dependent sets in M.
- B_M the bases of M, that is to say the maximal independent sets
- r_M, the rank function of M, it is defined as $r(X) = \max\{|Y| : Y \subseteq X, Y \in I\}$

These four definitions are all equal, this is easily shown.

- Having C_M, I_M is the set containing no set that is in C_M.
- Having B_M, I_M is the set being subsets of members of B_M
- Having r_M, $I_M = \{X \subseteq E : r_M(X) = |X|\}$

To get a better visual understanding of hereditary systems and matroids, this can be schematized as follows:

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|                 |
|                 |
|       D         |
|                 |
|........_________|
|  C    |   B     |
|--------.........|
|                 |
|       I         |
|                 |
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The set on the top is the dependent sets D, where the circuits C are the smallest dependent sets. On the bottom we have the independent sets I of which the largest are called the bases, B.

3.2 Defining the matroid.

The matroid is defined by a restriction on the set A_M. This has as an effect that there is a plethora of ways to define matroids, since these are derived from hereditary systems. If any of the following equivalent properties holds for a hereditary system, it is a matroid:

- weak absorption - $r(X) = r(X \cup \{e\}) = r(X \cup \{f\}) \Rightarrow r(X+e+f) = r(X)$ for $X \subseteq E, e, f \in E$.
- strong absorption - given two sets $X, Y \subseteq E$ and $\forall e \in Y.r(X+e) = r(x) \Rightarrow r(X \cup Y) = r(X)$.
- augmentation - if $I_1, I_2 \in I$ then $\exists e \in I_2 - I_1.I_1 \cup \{e\} \in I$, where $|I_2| > |I_1|$.
- base exchange - given two sets $B_1, B_2 \in B$ we have that $\forall e \in B_2 - B_1 \exists f \in B_1 - B_2.B_1 - \{e\} \cup \{f\} \in B$.
- uniformity - for all $X \subseteq E$, the maximal subsets of X belonging to I have the same size.
- weak elimination - given two circuits $C_1, C_2 \in C$ such that $x \in C_1 \cap C_2$ then $\exists C_3 \in C.C_3 \subseteq (C_1 \cap C_2) - \{x\}$.
- induced circuits - given $I_1 \in I$ then $I_1 \cup \{e\}$ contains at most one circuit.

There is also a greedy algorithm that given a hereditary system can show that it is a matroid, and as such is also a property of a matroid that is equivalent with the other properties.

Often the definition of a matroid is given as one of the sets that defines a hereditary system and one of the aspects, or properties, that defines the matroid. For example Wolfram's website presents two definitions, one with circuits and weak elimination, and one with independent sets and uniformity, the latter also used by Berge in his book *Hypergraphs*. The properties have all been used by different authors to define matroids, and the list is by no means exhaustive.

Apart from this, there is also several ways to construct new matroids from old ones, based on different sets of elements. We can for example restrict and contract a matroid, and we still have a matroid, or more precisely a matroid minor. A direct sum between matroids and a union between matroids does also remain a matroid.

There exists a lot of special matroids having certain properties, among them binary matroids, transversal matroids, graphic matroids, uniform matroids, vector matroids and coxeter matroids to mention a few.

Here is a presentation of some of the properties of a matroid on a cycle matroid.

3.3 The Cycle Matroid.

If C is the minimal graph cycles of a graph H, then the hereditary system on $E(H)$ whose circuits are C is a cycle matroid or a graphic matroid. All matroids that are equivalent to some cycle matroid is called graphic (but is not called cyclic). This defines also the bases, the independent sets and the rank function.

It's easily shown that any of the mentioned properties holds for a graphic matroid. Here is a few:

3.3.1 Base exchange property.

The bases of a cycle matroid is the spanning trees of the maximal forests in G. As a consequence they are all equally sized, a necessary condition for a hereditary system, the only difference being the edges not included in the spanning trees. If we take an edge $e - B_2 \cap B_1$ where $B_1, B_2 \in B$ and removes it (from B_1), then the spanning tree is divided into two. But since B_2 also is a spanning tree we can take an edge in $B_2 - B_1$ to reconnect the tree.

3.3.2 Augmentation.

The independent sets I of a cycle matroid is the spanning sub-graphs of a graph G. Take one spanning sub-graph with edges in I_1. Let the number of components in I_1 be k, then $|I_1| = n - k$. Further the largest forest has $n - k = |I_1|$ edges. As a consequence I_2 has an edge that connects two components in I_1, and thus $e \cup I_1$ is a larger independent set.

3.3.3 Weak elimination.

The circuits of $M(G)$ is the minimal cycles of the graph, and if we have two circuits C_1, C_2 they can thus not be contained in each other. They might have an edge e in common. If $e \in C_1 \cap C_2$ then there exists $C_3 \in C$ such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$. This is probably most easily understood if thinking graphically. The weak elimination property captures that no circuit is completely contained in another.

3.4 Two theorems.

The intersection of two hereditary systems defined on a Graph E is $\{X \subseteq E : X \in I_1 \cap I_2\}$ where I_1, I_2 is the independent sets of the hereditary systems. Edmonds presented a theorem 1970 that has unified several other theorems, and can be showed equivalent to the matroid union theorem.

Theorem 3.1 *For matroids M_1, M_2 on E the size of a largest common independent set satisfies*

$$\max\{|I| : I \in I_1 \cap I_2\} = \min_{X \subseteq E}\{r_1(X) + r_2(\bar{X})\} \tag{5}$$

4 back to hypergraphs...

A simple matroid can also be represented by a, and is a, hereditary hypergraph, the name maybe gives a hint. We can conclude some facts easily: given a matroid M on an underlying set E, with bases B of rank $r(M) = |B|$, then the bases can be represented as a $r(M)$-uniform hypergraph. Further, the sub-hypergraph of the hypergraphic representation of the matroid is also a matroid of rank $r(M)$.

A cycle matroid $M(G)$ having as bases the maximal spanning trees of G, is thus also representable as a hereditary hypergraph or even simpler, as a uniform hypergraph of rank $|I|$, since we know that a matroid is independently described by all its aspects.

There is a lot of results regarding hypergraphs and matroids that are not to difficult to derive, and it is interesting enough that a lot of results on hypergraphs are also valid for matroids, since they are a special case of hypergraphs.

4.1 The hypergraphic matroid

There exists also a generalization of the cycle matroid to a hypergraphic matroid, and this was first presented by a certain Lorea in 1975. I have not read that paper, but it is referred to in an article from 2005 by Frank. et. al. where it is used to give a generalization of Tutte's disjoint trees theorem. A proof is supplied for the hypergraphic matroid, that first defines hypercircuits, and then shows how they satisfy the weak elimination property.

5 Conclusion

The hypergraphs gives us a more compact notion for describing various problems in traditional graph theory, just like matroid theory. In both cases several in traditional theorems has been unified together in a more compact fashion. These two subjects interacts with each other, and in some cases they are equivalent. Both subject act outside the arena of graph theory as well, having applications in for example combinatorics and linear algebra.

I have jumped quite wildly in this rapport, but I wanted to show a bit of each subject, and also show how close to each other hypergraphs and matroids actually are, without going to deeply into the details.

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