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GRAPH MINOR THEORY

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Abstract. A monumental project in graph theory was recently completed. The project, started by Robertson and Seymour, and later joined by Thomas, led to entirely new concepts and a new way of looking at graph theory.

The motivating problem was Kuratowski's characterization of planar graphs, and a far-reaching generalization of this, conjectured by Wagner: If a class of graphs is minor-closed (i.e., it is closed under deleting and contracting edges), then it can be characterized by a *finite* number of excluded minors. The proof of this conjecture is based on a very general theorem about the structure of large graphs: If a minor-closed class of graphs does not contain all graphs, then every graph in it is glued together in a tree-like fashion from graphs that can almost be embedded in a fixed surface.

We describe the precise formulation of the main results and survey some of its applications to algorithmic and structural problems in graph theory.

1. Introduction

Let us start with recalling Kuratowski's Theorem [12] (see Figure 1):

Theorem 1. *A graph G is embeddable in the plane if and only if it does not contain a subgraph homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$.*

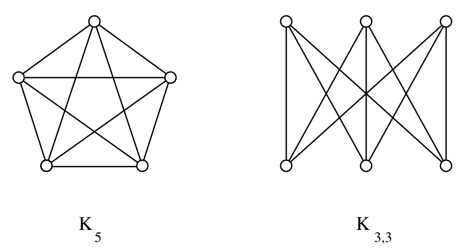

Figure 1. Excluded minors for planar graphs.

It is an immediate and natural question to ask if a similar result holds for other surfaces: can one characterize graphs embeddable in a fixed surface Σ by excluding subgraphs homeomorphic to graphs in a finite list? Studies concerning specific surfaces are somewhat discouraging: it seems that the only surface besides the

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plane (or sphere) for which such a list (of 35 graphs) is known is the projective plane. Nevertheless, the existence of a finite list was proved for the non-orientable case by Archdeacon and Huneke [1] and for the general case by Robertson and Seymour [19].

Klaus Wagner formulated a fundamental conjecture (apparently published only in 1970 in a textbook [36]), which extends this finite basis property to a much more general setting, namely to *minor-closed* classes of graphs. This conjecture was proved by Robertson and Seymour in a series of papers; the final version of the paper in which the proof is completed has just been finished. This gives us the excuse to survey this monumental work.

A crucial element of the proof is a theorem about the structure of graphs not containing a certain minor. Roughly speaking, it says that if a graph does not contain a certain minor, then it is basically 2-dimensional. The exact statement of the theorem (section 5) will be more complicated.

2. Minors and embeddings

Unless we state it otherwise, *graph* means a finite undirected graph in which parallel edges and loops are allowed. Given such a graph G, we consider the following three ways of reducing it:

(a) delete an edge,

(b) contract an edge,

(c) delete an isolated node.

Any graph G' that can be produced from G by successive application of these reductions is called a *minor* of G. (In particular, G is a minor of itself.) Every graph that is isomorphic to a minor of G is also called a minor of G. A minor that is not isomorphic to G is called a *proper* minor.

This notion fits well with many notions and problems graph theory studies. In fact, if a graph theorist learns about a property that is inherited by minors, he or she knows that this property is almost surely interesting from a graph theoretical point of view.

Let us see some simple but important graph properties that are minor closed (inherited by minors). Being cycle-free (i.e., a forest) is one. Being *series-parallel* is a more complicated example: these are graphs that can be obtained from a single edge by a sequence of parallel extensions (adding an edge parallel to an edge that already exists) and series extensions (subdividing an edge by a new node).

Various topological properties of graphs are also often minor-closed. Planarity of graphs is an example. We can generalize this: the property of being embeddable in any fixed surface is inherited by minors.

Every graph is embeddable in $\mathbb{R}^3$, but we may impose additional conditions on such embeddings. For example, the graph is *linklessly embeddable* if it has an embedding in which no two disjoint cycles of the graph are linked. This may be defined in various ways: We may require that their linking number is 0 or that their modulo 2 linking number is 0 or that they are homotopically unlinked in the sense that they can be continuously deformed without ever intersecting each other so that they end up on different sides of a plane. For a given embedding, these are different notions; however, it is a nontrivial consequence of the results below that they are equivalent from the point of view of the existence of an embedding.

For example, if a graph is embeddable in $\mathbb{R}^3$ so that the linking number of any two cycles is even, then we can modify the embedding so that any two cycles will become homotopically unlinked.

A similar notion is *knotlessly embeddable*: these graphs have an embedding in 3-space in which no two cycles are linked and no cycle is knotted. Both of these topological properties are minor-closed.

3. Wagner's conjecture

3.1. **Excluded minor characterizations.** Many important and deep theorems characterize minor-closed graph properties by "excluded minors". Let us start with simple examples though. Clearly *a graph is a forest if and only if it does not contain the triangle K_3 as a minor.* Dirac [9] proved:

Theorem 2. *A graph is series-parallel if and only if it has no K_4 minor.*

Kuratowski's Theorem 1 is not quite of this form, but Wagner [35] reformulated it in this way: he showed that instead of excluding K_5 and $K_{3,3}$ as subgraphs up to homeomorphism, it is equivalent to exclude them as minors.

A much more difficult theorem of this type was conjectured by Sachs and proved by Robertson, Seymour and Thomas [28]:

Theorem 3. *A graph is linklessly embeddable if and only if it does not contain any of the seven graphs in Figure* 2 *as a minor.*

No such theorem is known for knotlessly embeddable graphs, even though the main result to be discussed implies that a finite family characterizing them does exist.

3.2. **Statements of the theorem.** We say that a class $\mathcal{K}$ of graphs is *minor-closed* if for every $G \in \mathcal{K}$ every minor of G also belongs to $\mathcal{K}$.

Given a family of graphs $\{G_1, G_2, \dots\}$, we can consider the class $\mathcal{K}$ of graphs that do not contain any of $G_1, G_2, \dots$ as a minor. Trivially, this class is minor-closed; we'll say that the graphs $G_1, G_2, \dots$ *characterize $\mathcal{K}$ as excluded minors.* It is also trivial that every minor-closed family can be characterized by excluded minors: just list all graphs not in the family. Wagner's conjecture (now the Robertson–Seymour Theorem) asserts that we can always achieve this by a finite list.

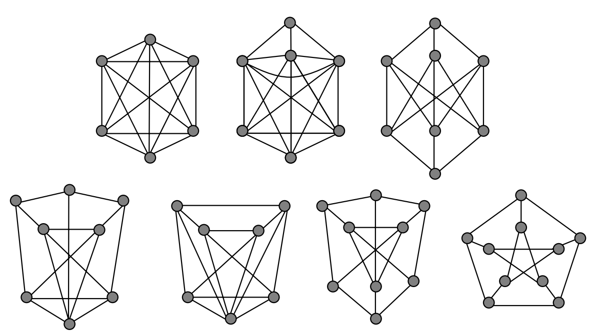

FIGURE 2. Excluded minors for linklessly embeddable graphs (the "Petersen family").

Theorem 4. *Every minor-closed class of graphs can be characterized by a finite family of excluded minors.*

Clearly this theorem is a far-reaching generalization of Kuratowski's Theorem.

For every minor-closed class $\mathcal{K}$ there is a unique *minimal* list of excluded minors characterizing it: this consists of those graphs not in $\mathcal{K}$ for which every proper minor is in $\mathcal{K}$. Theorem 4 asserts that the set of minor-minimal graphs not in $\mathcal{K}$ is finite.

Yet another formulation of this result is that *in every infinite set $\{G_1, G_2, \dots\}$ of finite graphs there are two graphs such that one is a minor of the other.* This form puts it in the context of the quite extensively studied theory of well-quasi-ordering (see e.g. [11]). A partially ordered set $(P, \leq)$ is called *well-quasi-ordered* if every infinite sequence $(x_1, x_2, \dots)$ of its elements has two elements x_i and x_j such that $i < j$ and $x_i \leq x_j$. It is easy to see that this is equivalent to saying that $(P, \leq)$ contains neither an infinite descending chain nor an infinite antichain (mutually incomparable elements). Theorem 4 says that the set of (isomorphism classes of) finite graphs, with the "minor" relation as partial order, is well-quasi-ordered. In this case it is trivial that no infinite descending chain exists, so well-quasi-ordering means that there is no infinite antichain.

3.3. **Trees.** There are results on well-quasi-ordering that can be viewed as special cases of the Wagner conjecture. Let T_1 and T_2 be two (finite) rooted trees. We say that $T_1 \leq T_2$ if there is a subdivision of T_1 that can be embedded into T_2 so that the root of T_1 is mapped onto the root of T_2.

As early as 1937, Vázsonyi formulated the following conjecture, which was proved in 1960 independently by Kruskal [10] and Tarkowski [31]:

Theorem 5. *Finite trees are well-quasi-ordered with respect to the relation $\leq$.*

This theorem is a bit stronger than Wagner's Conjecture specialized to trees. In the other direction, there are an unbounded number of generalizations, for example, to infinite trees, trees embedded in the plane, trees with nodes labeled from a finite ordered set, etc.

Perhaps the simplest proof of Theorem 5 is due to Nash-Williams [15], which we sketch here for a reason. Let us call an infinite sequence $(T_1, T_2, \dots)$ of rooted trees *bad* if there are no subscripts $i < j$ with $T_i \leq T_j$. Suppose that there is a bad sequence of trees, and let T_1 be the smallest tree that can start such a sequence. Continuing, we can select a sequence $(T_1, T_2, \dots)$ such that for each k, T_k is the smallest tree which occurs as the k-th element of a bad sequence starting with $(T_1, \dots, T_{k-1})$. It is easy to check that the sequence $(T_1, T_2, \dots)$ itself is bad.

Let T_i' be the tree obtained from T_i by deleting any branch from the root, and let T_i'' be the branch that is deleted (so T_i'' is rooted at a child of the root of T_i). The key observation is that the sequence $(T_1', T_2', \dots)$ cannot contain a bad subsequence. Indeed, if say $(T_{i_1}', T_{i_2}', \dots)$ were a bad subsequence, then $(T_1, \dots, T_{i_1-1}, T_{i_1}', T_{i_2}', \dots)$ would be a bad sequence, which would be a contradiction since T_i' is smaller than T_i. But if the sequence $(T_1', T_2', \dots)$ does not contain a bad subsequence, then it contains an infinite increasing subsequence $T_{j_1}' \leq T_{j_2}' \leq \dots$. Now the sequence $(T_{j_1}'', T_{j_2}'', \dots)$ cannot be bad (by the same argument as we had for the sequence $(T_{i_1}', T_{i_2}', \dots)$), and so there must be a $k < l$ such that $T_{j_k}'' \leq T_{j_l}''$. Together with $T_{j_k}' \leq T_{j_l}'$ this implies that $T_{j_k} \leq T_{j_l}$, which contradicts the fact that $(T_1, T_2, \dots)$ is bad.

One interesting point in this proof is that it is non-constructive: the critical bad sequence can be found only by using the Axiom of Choice.

3.4. **Embedding in a surface.** Perhaps the most important special case of Wagner's Conjecture concerns embeddability in a fixed surface. This had been conjectured for a long time; after many attempts, this special case was proved for the non-orientable case by Archdeacon and Huneke [1] and for the general case by Robertson and Seymour [19]:

Corollary 6. *For every closed compact surface there is a finite list of graphs such that a graph G is embeddable in this surface if and only if it does not contain any of these as a minor.*

It is not hard to see that an analogous theorem holds where a finite list of graphs are excluded as homeomorphic subgraphs (rather than minors). We'll see that this special case plays an important role in the proof of the general conjecture.

4. Related graph-theoretical problems

4.1. **Linkages.** There is an important graph theoretic problem that plays a central role in the theory. Given a graph and $2k$ nodes $s_1, \dots, s_k, t_1, \dots, t_k$, we may want to know whether there are k disjoint paths $P_1, \dots, P_k$ so that P_i connects u_i to v_i. If such paths exist, we say that the ordered sets $(s_1, \dots, s_k)$ and $(t_1, \dots, t_k)$ are *linked*. If every two ordered k-sets are linked, we say that the graph is *k-linked*.

The linkage problem sounds very similar to Menger's Theorem, which asserts that *for two k-element sets S and T, we have k disjoint paths, each connecting a node in S to a node in T if and only if S and T cannot be separated by $k-1$ nodes.* The additional condition that we prescribe which element of S should be connected to which element of T makes this problem much more difficult. A complete characterization exists only for $k = 2$ (Thomassen [32], Seymour [29]). Let us assume, to exclude some not-so-interesting complications, that the graph is 4-connected (i.e., it cannot be separated by 3 or fewer nodes).

Theorem 7. *Let G be a 4-connected graph and s_1, s_2, t_1, t_2 four nodes of G. Then (s_1, s_2) and (t_1, t_2) are linked unless G is planar and s_1, s_2, t_1, t_2 are on the boundary of the same face, in this cyclic order.*

It is interesting that the answer to a purely graph-theoretic question involves such a strong topological property of the graph.

The linkage problem is very important in many applications: it plays a crucial role in VLSI design and is closely related to the Multicommodity Flow Problem in discrete optimization.

Linkages are "linked" to graph minors in a number of ways. To illustrate the idea, let us first consider a graph G that is k-linked. Let H be a graph with k edges. Then we can find a homeomorphic copy of H in G by first mapping the nodes of H arbitrarily, specifying the edges through which the connecting paths should start, and then solving a linkage problem to map the edges of H onto paths in G.

In the other direction, Robertson and Seymour [22] proved that if a graph is $2k$-connected and has a K_{3k} minor, then the graph is k-linked. Extending these ideas, Bollobás and Thomason [4] proved that every $(22k)$-connected graph is k-linked. For more on this connection, see [5].

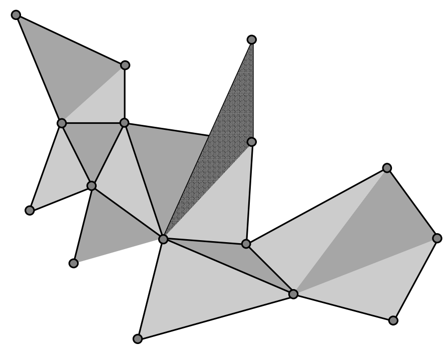

FIGURE 3. A graph with tree-width 2. The shaded triangles indicate the subgraphs G_i.

4.2. **Tree-decomposition and tree-width.** Let G_1 and G_2 be two graphs, and let $S_i \subseteq V(G_i)$ be a k-clique (a set of k mutually adjacent nodes). Let G be obtained by identifying S_1 with S_2 and deleting some (possibly none, possibly all) edges between the nodes in $S_1 = S_2$. We say that G is a *k-clique-sum* of G_1 and G_2.

The following notion is crucial in the theory of graph minors. Suppose that we can write our graph G as the union of subgraphs G_i, which are indexed by the nodes of a tree T, with the following property: if i, j, k are three nodes of T and j lies on the path between i and k, then $V(G_i) \cap V(G_k) \subseteq V(G_j)$. Then we say that the family $(G_i : i \in V(T))$ is a *tree-decomposition* of the graph G. The graphs G_i are called the *parts* of the tree-decomposition.

The *tree-width* of a graph G is the smallest integer k such that G has a tree-decomposition into parts with at most $k+1$ nodes. Equivalently, G can be obtained by repeatedly taking clique-sum with graphs with at most $k+1$ nodes (Figure 3). A graph with tree-width 1 is a forest.

5. STRUCTURE THEORY

5.1. **Constructive characterizations.** Let us fix a graph H and consider the class $\mathcal{K}_H$ of graphs not containing H as a minor. It is clear that this class is in NP (to certify that a graph G is not in $\mathcal{K}_H$, just exhibit the way H is produced from G as a minor). It follows from Graph Minor Theory that this class is in P, and so also in co-NP. Ignoring the issue of a polynomial time test for the time being, let us ask: How can we certify that $G \in \mathcal{K}_H$, i.e., that G does not contain H as a minor?

As an illustration, let us quote Wagner's characterization of graphs not containing the complete graph K_5 as a minor [34]. We denote by V_8 the graph obtained from a cycle of length 8 by connecting opposite nodes.

Theorem 8. *A graph G has no K_5 minor if and only if it can be obtained by* 0-*,* 1-*,* 2- *and* 3-*clique-sum operations from planar graphs and* V_8.

This theorem can serve as a paradigm for answering such a question: we find a class that does not contain a K_5 minor for topological reasons (planar graphs), throw in some exceptions, and describe gluing rules that preserve the property that there is no K_5 minor. But this theorem also warns us that such a certificate

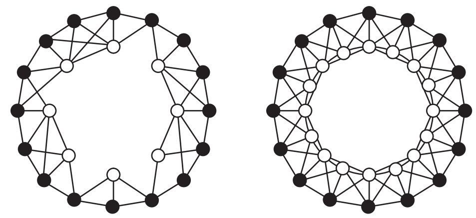

FIGURE 4. A fringe of width 2 and a fringe of width 3.

can become quite complicated, and in general it is probably hopeless to explicitly describe the basic classes and gluing rules that would produce $\mathcal{K}_H$.

5.2. **Approximate characterizations.** The main idea behind a successful structure theory for graphs with excluded minors is to prove such a result in an approximate sense. We start with an early result of this kind from [18].

Theorem 9. (a) *For every planar graph H there is an integer $k > 0$ such that if a graph does not contain H as a minor, then its tree-width is at most k.*

(b) *For every $k > 0$ there is a planar graph H such that no graph with tree-width at most k contains H as a minor.*

Part (b) is easy: a sufficiently large square grid does the job. The main assertion here is that if a graph does not contain a given planar graph H as a minor, then it has bounded tree-width, and therefore it can be constructed from bounded size graphs by gluing them together in a tree-like structure.

Robertson and Seymour [23] give an analogous (but much more difficult) construction that describes the approximate structure of every graph in $\mathcal{K}_H$ (even though it does not characterize $\mathcal{K}_H$).

We need a definition. Let C be a cycle. Select a family of arcs on C so that each node is contained in at most k of these arcs. For each arc A, create a new node v_A. Connect v_A to some nodes on the arc A. Also connect any number of pairs (v_A, v_B) for which A and B have a common node. We call this *adding a fringe of width k* to C (see Figure 4).

For a positive integer k, construct the following class $\mathcal{L}_k$ of graphs:

(i) We start with a graph G embedded in a connected closed surface Σ with genus at most k so that each face is homeomorphic with an open disc.

(ii) We select at most k faces of G and add a fringe of width at most k to each of them.

(iii) We create at most k new nodes and connect them to the other nodes arbitrarily.

(iv) We repeatedly construct the k-clique-sum of the graph we have with another graph constructed using steps (i)–(iii) above.

It is clear that the class $\mathcal{L}_k$ is in NP: to certify that a graph is in $\mathcal{L}_k$, just follow the construction. The assertion that this construction provides an "approximate good characterization" of classes characterized by excluded minors is made precise by the following fundamental theorem [24]:

Theorem 10. *(a) For every graph H there exists an integer $k > 0$ such that $\mathcal{K}_H \subseteq \mathcal{L}_k$;*

(b) For every integer $k > 0$ there exists a graph H such that $\mathcal{L}_k \subseteq \mathcal{K}_H$.

The second assertion is not hard, and it is included here just for completeness. The hard, and useful, part is (a). One can strengthen it in various ways—for example, in (i) we may start with a surface on which H does not embed.

6. About the proofs

6.1. **The Structure Theorem.** The proof of the Structure Theorem is difficult and uses much of the theory (linkages, tree-width, etc.). We'll confine ourselves to a hint about the puzzling question of how topology comes in at all; in other words, how do we get from a condition that is purely combinatorial (i.e., an excluded minor) any information about embeddings?

Theorem 7 may provide such a hint. If we find disjoint paths between certain pairs of nodes in a certain part of the graph, then we can use these to construct appropriate minors. If not, then we know that this part of the graph is planar, which could be the beginning of an embedding of the whole graph.

6.2. **Wagner's Conjecture.** Let us try to extend the proof of Vazsonyi's conjecture sketched above. If Wagner's Conjecture is not true, then we can select an infinite sequence $(G_1, G_2, \dots)$ of graphs such that G_i is not a minor of G_j for $i < j$. We may repeat the argument there and make each G_k a minimal continuation of the previously selected sequence.

At this point, we are stuck: for trees, we could decompose the graphs into two parts and apply a simple argument separately to the parts. But for a general graph, no such decomposition offers itself.

Here is where the results in Section 5 come into play: no G_i with $i > 1$ contains G_1 as a minor, so they are not just arbitrary graphs, but graphs with a special structure!

First, suppose that G_1 is planar. Then we know by Theorem 9 that there is an integer $k > 0$ (depending only on G_1) such that all the G_i $(i > 1)$ have tree-width at most k. So Wagner's conjecture follows in this case if we show that *for any positive integer k, the class of graphs with tree-width at most k is well-quasi-ordered by the minor relation.* This assertion is very substantially more difficult than Theorem 5, but still it is in the same spirit and can be proved by somewhat similar means [18].

Second, suppose that G_1 is non-planar. In this case we have to use the difficult and complicated Structure Theorem 10. This gives a 2-layer structure: outside, we have a tree-like structure (which can be handled by the methods we already saw), but inside we have graphs that are (approximately) embedded in a given surface. So while the notion of a minor-closed class is substantially more general than the notion of graphs embeddable in a given surface, Theorem 10 reduces Theorem 8 to just this special case, namely Corollary 6. To be more precise, since Theorem 10 gives only an approximate embedding in a surface, Robertson and Seymour have to encode the extra structure (the "fringes") into a hypergraph and then extend Corollary 6 to hypergraphs embedded in a surface.

7. Algorithmic consequences

Graph minor theory has an algorithmic consequence that is unprecedented in its generality [22].

Theorem 11. *Every minor-closed property of graphs can be tested in polynomial time.*

The algorithm that follows from the Graph Minor Theory is of complexity $O(n^3)$. The devil is hidden in the big-O: first, the constants are huge; and second, they depend on the list of excluded minors. While the finiteness of this list is guaranteed by Theorem 8, it is in general not easy to find and, as we have remarked, it can be very large. So (unless the property that we want to test is given by an explicit list of excluded minors), Theorem 11 tells us only the existence of a polynomial time algorithm; it is a very unusual "pure existence theorem" for an algorithm.

The notion of tree-width introduced above has turned out to be quite useful in algorithm design: there are many graph parameters that are difficult (NP-hard) to compute in general but that can be computed in polynomial time if the graph in question has bounded tree-width. In view of Theorem 9, this means that if we restrict our attention to graphs not containing a given planar graph as a minor, then we can solve many problems in polynomial time that are NP-hard in general (see e.g. [2]).

Let the chromatic number serve as an example: this is defined as the minimum number of colors needed to color the nodes of a graph G so that adjacent nodes get different colors. This fundamental parameter is difficult (NP-hard) to compute, but for graphs with bounded tree-width we can use the following method.

Let G be a graph with tree-width at most k; we want to decide whether it is colorable with r colors. We know that G can be glued together from pieces G_i with at most $k+1$ nodes, which are indexed by the nodes of a tree T, satisfying (TW1) above. Let us designate one of these G_i, say G_1, as the *root region.* Our algorithm will be recursive and in fact do more: it will decide for every r-coloring of the root region whether or not the coloring can be extended to a (legitimate) r-coloring of the whole graph.

Now the algorithm consists of two easy recursive steps:

(1) If the root region, as a node of T, has degree $d > 1$, we decompose the tree into its "branches" relative to the root. These branches correspond to subgraphs of G for which the extension problem can be solved recursively. A coloring of the root can be extended to G if and only if it can be extended to every branch.

(2) If the root region has degree $d = 1$ in T, we delete this node from T and designate its neighbor as the new root region. We solve the extension problem for this smaller graph, and it is easy to check which colorings of the old root can be extended to the new root and which of these can be extended to the rest.

8. The decomposition paradigm

The excluded minor characterizations and the structure theorems discussed above can serve as prototypical examples of a paradigm that leads to many difficult but important results.

Perhaps most dramatic of these is the recent resolution of the Strong Perfect Graph Conjecture by Chudnovsky, Robertson, Seymour and Thomas [6]. Here again, the key to the proof is a structure theory which describes how every perfect graph can be glued together from certain basic types. The minor-producing operation in this case is deleting a node.

The paradigm goes way beyond graph theory. A beautiful and rather early example is a pair of deep theorems on regular matroids. These are matroids that can be coordinatized by a totally unimodular matrix. Interest in them comes from the fact that two standard matroids derived from graphs, the cycle matroid and the cocycle matroid of a graph, are totally unimodular. The question of characterizing regular matroids is closely related to (but not quite equivalent with) characterizing totally unimodular matrices. Tutte [33] gave a characterization in terms of excluded minors. On the other hand, Seymour [30] gave a constructive description of regular matroids: he showed that they can be glued together (in a way analogous to 1- and 2- and 3-clique-sums) from cycle matroids and cocycle matroids of graphs, and one particular 10 element matroid. Tutte's result tells you why a matroid is *not* regular: it is because it contains, as a minor, one of three particular matroids. Seymour's result tells you why a matroid is regular: because it can be built up in a specific way.

We should also mention the characterization of balanced matrices using the same paradigm by Conforti, Cornuejols, Kapoor, Rao and Vuskovic [7, 8].

9. Research directions

9.1. **Simpler proofs.** It would be quite important to have simpler proofs with more explicit bounds. Warning: many of us have tried, but only a few successes can be reported. For the generalization of Kuratowski's Theorem to other surfaces (Corollary 6) such proofs are known: Archdeacon and Huneke [1] proved it (before the Robertson-Seymour proof of the general result) for the non-orientable case, and Mohar [13] gave a constructive proof for the orientable case.

9.2. **Exact structural descriptions.** If a class of graphs is defined in terms of excluded minors, then it is in co-NP (it is easy to certify that a graph contains one of these). We also know that it is in P, and hence, also in NP; but is there a direct way to certify that a graph is in this class? A structure theorem could serve this purpose (an example would be Wagner's Theorem 8), but such structure theorems are known only in special cases, and in the general case, we have only the approximate structure theorem, Theorem 10. (This should also warn us that such a result could be very difficult.)

9.3. **Properties of minor-closed classes.** It seems that there are many interesting known and unknown general properties of minor-closed classes; some follow from Theorem 10, others need (or should need) different techniques. To give an example: an old result of Babai [3] asserts that *if $\mathcal{K}$ is a minor-closed class of graphs that does not contain all graphs, then graphs in $\mathcal{K}$ cannot have arbitrary automorphism groups.*

We have seen above that if we restrict our attention to graphs that do not contain a given planar graph as a minor, then many hard algorithmic problems become polynomially solvable. There are also several examples of hard algorithmic problems (for example, a version of the linkage problem in Section 4.1) that are polynomially solvable for planar graphs. In some cases, Theorem 10 allows us to extend these to any minor-closed class of graphs, but there are other such problems where this extension does not seem to work.

9.4. **3-dimensional graphs.** One way of interpreting Theorem 10 is that graphs that don't have all minors are essentially 2-dimensional, and vice versa. Is there a similar description of "3-dimensional" graphs? Is there a general notion of "minor" that would correspond to graphs whose structure we feel is 3-dimensional?

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