Small Survey on Perfect Graphs

Michele Alberti
ENS Lyon

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Abstract

This is a small survey on the exciting world of Perfect Graphs. We will see when a graph is perfect and which are families of graphs that are perfect. The notion of perfect graph is due to the French mathematician Claude Berge. He formulated two conjecture on perfect graphs, and the stronger one stood for 40 years.

1 What is it a perfect graph?

The first formalization of "perfect graph" is due to the French mathematician Claude Berge in 1963, even if there were some other studies in that direction before.
A starting point to introduce the rich family of perfect graphs is the notion of chromatic number $\chi(G)$ of a graph G. This is the minimum number of colors needed to color all the vertices of a graph, so that every two adjacent vertices get different colors. In other words, the minimum number of colors needed to properly color the graph G. Let's reflect about it.
It's well known that determining the chromatic number of a graph is a hard problem, NP-hard, and seems to be also very hard to get lower bounds. Obviously, given a graph G, one lower bound is $\omega(G)$, the size of the biggest complete graph (or clique) of G. Why? Well, it's pretty easy to see. If a graph contains four pairwise adjacent vertices, surely we need at least four colors to color it.
There are some graphs for which $\chi(G)$=$\omega(G)$ holds. This is a first clue on the way that will bring us to talk about perfect graphs.

1.1 The importance of an inheritance condition

The condition just stated before, picked alone, is not enough to say that a graph is actually interesting in some senses. The reason is the following. Let's consider the complete graph of three vertices, K_3 (also called "triangle"), and the cycle of five vertices, C_5. Now take the disjoint union of the these two. The chromatic number of the obtained graph is three. Indeed, you need three

colors to color K_3 and the same quantity holds also for C_5. Moreover, the largest clique in the graph is the triangle. So the graph satisfies $\chi(G)=\omega(G)$, but it is definitely not interesting in general.
To avoid such graphs and, at the same time keep the interesting ones, Berge used the following method: make the $\chi(G)=\omega(G)$ property hereditary.

Definition 1. A graph G is **perfect** if $\chi(H)=\omega(H)$ for every induced subgraph H of G.

Thanks to this definition, we can "discard" graphs like the last considered because they are not perfect.

1.2 Imperfect graphs and minimal imperfect graphs

It's not hard to see that any odd cycle of length at least five is imperfect. Indeed, it holds that,

Proposition 1. If $t \geq 2$, then $\chi(C_{2t+1}) > \omega(C_{2t+1})$.

Proof. First of all, observe that any such graph G has $\omega(G) = 2$. So, at least, we need two colors. Let's assign a numeration, from 1 to $2t+1$, to the vertices of the graph and then, assign the first color to odd vertices and the second one to even vertices. In this way, we know that two adjacent vertices have different colors. But now, the vertex 1 and the vertex $2t+1$ are adjacent and they are odd. So we need a third color, thus, $\chi(G) = 3$. □

Actually, the proposition holds also for the complement of such graphs. That is, $\chi(\bar{C}_{2t+1}) > \omega(\bar{C}_{2t+1})$.
The following is a special family of imperfect graphs.

Definition 2. A graph G is minimally imperfect if G is not perfect, but all its proper induced subgraphs are.

A classical example is C_5 and it is also the smallest of this family (obviously, its complement too because they are the same graph).

Lemma 1. *In a minimal imperfect graph, no stable set intersects every maximum clique.*

Proof. Consider G a minimal imperfect graph. Now suppose that there exists a stable set S intersecting every maximum clique of size $\omega(G)$. Consider $G-S$, it is smaller than G and then, by definition of minimal imperfect graph for G, is perfect. By definition of perfection, $\chi(G-S) = \omega(G-S)$. Since S intersects every maximum clique at least with one vertex, by hypothesis, and at most with one, otherwise it cannot be a stable set, it is clear that $\omega(G-S) = \omega(G)-1$. Since $G-S$ is perfect, it can be colored with $\omega(G)-1$ different colors. But now, it easy to see that G can be properly colored. Since

S intersects all the maximum clique, it is enough to color the vertices of S with a new color to obtain a $\omega(G)$-coloring of the graph. So we have that $\chi(G) = \omega(G)$. Now, by definition of G, each its proper induced subgraph H is perfect and thus $\chi(H) = \omega(H)$. Thus G is perfect, contradicting the hypothesis. $\square$

As we have discussed before, there exists an imperfect graph G that verifies $\chi(G) = \omega(G)$. The problem of its imperfection comes from the fact that there exists at least one of its proper induced subgraph that it does not inherit this property. It is nice to determine the smallest connected imperfect graph with this property.

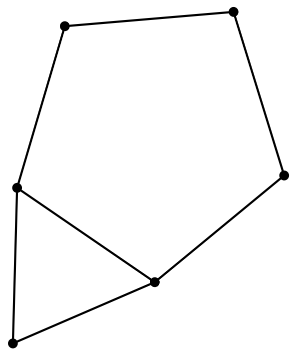

Figure 1: The graph G exhibits $\chi(G) = \omega(G)$, but it's not perfect

Proposition 2. The graph G in figure 1 is the smallest connected imperfect graph such that $\chi(G) = \omega(G)$.

Proof. First of all, observe that the maximum clique in G has size three and we need three colors to color the entire graph. So $\chi(G) = \omega(G)$. It's imperfect because of the induced subgraph C_5. Let's call it H. Now, H is the smallest minimal imperfect graph with $\chi(H) = 3$ and $\omega(H) = 2$. So to adjust the latter value to the former, we have to add a triangle to H, but avoiding to insert a chord (otherwise, H will become perfect). It is clear that to use the least total number of verteces, a good way is to attach a side of the triangle to a side of H so that they share two vertices. But now, $\chi(H) = \omega(H)$ and $H = G$. $\square$

The reader should beware of the fact that we have not proved that given a imperfect graph G, then there must exists in G an induced subgraph that is an odd cycle (or its complement) of length at least five. This is a much stronger result, first conjectured by Berge.

1.3 Berge's conjectures

Let's first recall some standard parameters of undirected graphs that will be often used through all the report. Given an undirected graph G, we will identify:

- $\omega(G)$ as the *clique number* of G: the size of the largest complete subgraph of G.
- $\chi(G)$ as the *chromatic number* of G: the fewest number of colors needed to properly color the vertices of G.
- $\alpha(G)$ as the *stability number* of G: the size of the largest stable set of G.
- $k(G)$ as the *clique cover number* of G: the fewest number of complete subgraphs needed to cover the vertices of G.

Since the operation of graph complementing converts clique to stable sets and vice versa, it holds that $\omega(\bar{H})=\alpha(H)$, for any graph H. Moreover, properly coloring $\bar{H}$ means that the vertices of H can be expressed as a union of cliques in H. In this case, such set of cliques is called a clique covering of H. In his works, Berge defined two types of perfection for a graph G:

1. γ-perfection (he used $\gamma(G)$ for chromatic number), G is γ-perfect if $\chi(G[A])=\omega(G[A])$, for all $A \subseteq V(G)$.
2. α-perfection, G is α-perfect if $k(G[A])=\alpha(G[A])$, for all $A \subseteq V(G)$.

where, for the sake of completeness, $G[A]$ means the subgraph induced by the set of vertices A. Our definition seems to not consider α-perfection, but we will see that it does.
In 1961, Berge proposed two conjectures about perfect graphs:

- (The Weak Perfect Graph Conjecture, later Lovász theorem) A graph G is γ-perfect if and only if G α-perfect.
- (The Strong Perfect Graph Conjecture) A graph G is perfect if and only if G is *Berge*, that is, it contains no odd hole or antihole[1].

2 The Weak Perfect Graph Theorem

By duality, it is clear that a graph G is γ-perfect if and only if $\bar{G}$ is α-perfect. Moreover, Lovász[2] proved the first conjecture made by Berge on perfect

[1] A *hole* means an induced subgraph which is a cycle of length at least four, and an *antihole* is the complement.

[2] He shocked the world of combinatorics at the age of 22!

graphs. Thus, the following is another way to state Berge's first conjecture: "G is perfect if and only if $\bar{G}$ is perfect", w.r.t. our first definition for perfect graphs.
Now, we are going to show a graph manipulation that enlarges the size of graph preserving its γ-perfection and α-perfection.

Definition 3. Duplicating a vertex x of G produces a new graph $G' = G \circ x$, by adding a new vertex x' and then fastening it to the neighborhood of x ($N(x') = N(x)$).

Obviously, this operation can be iterated.

Definition 4. The vertex multiplication of G by the non negative integer vector $h = (h_1, \ldots, h_n)$ is the graph $H = G \circ h$ whose vertex set consists of h_i copies of each $x_i \in V(G)$, where copies of x_i are adjacent to x_j in H if and only if x_i is adjacent to x_j in G.

Let's see an example of vertex duplication. In figure 2, there is a generic graph G. By duplicating the vertex $x5$, the set of vertices is enlarged by one, $x5'$, and the latter vertex is fastened to the neighborhood of $x5$ (figure 3). Note how the operation does not enlarge any clique.

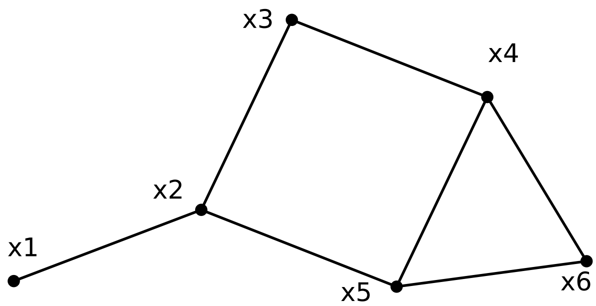

Figure 2: A graph G

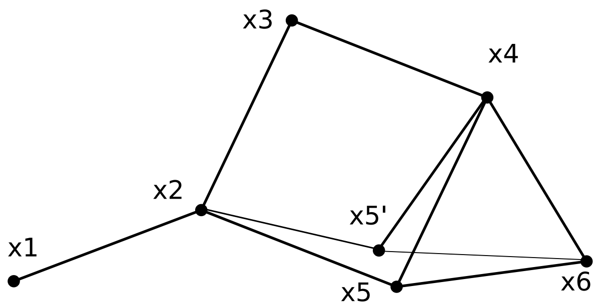

Figure 3: Vertex duplicating: $G \circ x5$

Remark 1. The definition of vertex multiplication permits $h_i = 0$, in which case the final graph is obtained from G by removing x_i. Thus, every induced subgraph of G can be obtained by an appropriate vertex multiplication with a (0,1)-valued vector.

In figure 4, there is an example of vertex multiplication.

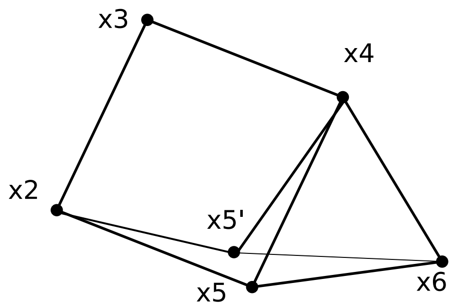

Figure 4: Vertex multiplication: $G \circ (0, 1, 1, 1, 2, 1)$

Lemma 2. *Vertex multiplication preserves γ-perfection and α-perfection.*

Proof. First of all, observe that $G \circ h$ can be obtained from an induced subgraph A of G by iterating vertex duplications. Starting from $G[A]$, we obtain $G \circ h$ by iterating vertex duplication of x_i up to h_i times, for all i. Indeed, each vertex duplication preserves the property that every copy of x_i is adjacent to x_j if and only if x_i is adjacent to x_j in G.
If G is α-perfect but $G \circ h$ is not, then there must be at least one operation of vertex duplication that from $G[A]$ produces a graph that is not α-perfect from an α-perfect one. The same holds for γ-perfection. So, observing that every proper induced subgraph of $G \circ x$ is either an induced subgraph of G or a vertex duplication of an induced subgraph of G, it suffices to prove that $\chi(G \circ x) = \omega(G \circ x)$ when G is γ-perfect and that $\alpha(G \circ x) = k(G \circ x)$ when G is α-perfect.
Case 1.(G is γ-perfect): this is the easy case. As observed before, duplicating a vertex does not enlarge any clique. But then, it's easy to extend a proper coloring of G to a proper coloring of $G \circ x$ by giving the color of x to x'. Since x and x' are disjoint, by definition of vertex duplication, no clique contains both and so $\omega(G \circ x) = \omega(G)$.
Hence, $\chi(G \circ x) = \chi(G) = \omega(G) = \omega(G \circ x)$.
Case 2.(G is α-perfect): we have to consider two cases.

1. x belongs to a maximum stable set in G. In this case, duplicating x enlarges the maximum stable set where x belongs to and then $\alpha(G \circ x) = \alpha(G) + 1$. Since $k(G) = \alpha(G)$, it's easy to obtain a clique covering of this size by adding x' as a 1-vertex clique to some set of (G) cliques covering G.

2. x does not belong to any maximum stable set in G. In this case, duplicating x does not enlarge the size of any maximum stable set, and so $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G. Since $k(G) = \alpha(G)$, Q intersects every maximum stable set of G. Now, since x doesn't belong to any maximum stable set, $Q' = Q - x$ also intersects every maximum stable set. We obtain that $\alpha(G - Q') = \alpha(G) - 1$. Now, by α-perfection of G, the induced subgraph $G - Q'$ is α-perfect.

   Consider $Q' \cup x'$. We know that it's a clique[3] and then, by adding it to a set of $\alpha(G) - 1$ cliques covering $G - Q'$ yields a set of $\alpha(G)$ cliques covering $G \circ x$.

□

Let's show the main theorem of this section, actually of the entire report.

Theorem 1 (Lovász 1972)**.** *A graph is perfect if and only if its complement is perfect.*

Proof. It suffices to show that α-perfection of G implies γ-perfection of G because then, applying this to $\bar{G}$ yields the converse. Suppose that the claim fails. Then we consider a minimal graph G that is α-perfect but not γ-perfect. But then, G is a minimal imperfect graph. By the lemma 1, we may assume that every maximal stable set S in G misses some maximum clique $Q(S)$. Now, we will design a special vertex multiplication of G.
Let $\mathbf{S} = \{S_i\}$ be the list of maximal stable set of G. We weight each vertex by its frequency in $\{Q(S_i)\}$, letting h_j be the number of the stable sets $S_i \in S$ such that $x_j \in Q(S_i)$. By the lemma just proved, $H = G \circ h$ is α-perfect, yielding $\alpha(H) = k(H)$. Now, we will use counting arguments for $\alpha(H)$ and $k(H)$ to obtain a contradiction.
Let A be the 0,1-matrix of the incidence relation between $\{Q(S_i)\}$ and $V(G)$; we have that $a_{i,j} = 1$ if and only if $x_j \in Q(S_i)$. By construction, h_j is the number of 1s in column j of A, and $n(H)$ is the total number of 1s in A. Since each row has $\omega(G)$ 1s, also $n(H) = \omega(G)\,|\mathbf{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) = \omega(G)$. Therefore, $k(H) \geq n(H)/\omega(H) = |\mathbf{S}|$.
We obtain a contradiction by proving that $\alpha(H) < |\mathbf{S}|$. Now, every stable set in H consists of copies of elements in some stable set of G, so a maximum stable set in H consists of all copies of all vertices in some maximal stable set of G. Hence $\alpha(H) = max_{T \in \mathbf{S}} \sum_{i:\, x_i \in T} h_i$. The sum counts the 1s in A that appear in the columns indexed by T. If we want to count these 1s instead by rows, we obtain $\alpha(H) = max_{T \in \mathbf{S}} \sum_{S \in \mathbf{S}} |T \cap Q(S)|$. Since T is a stable set, it shares at most one vertex with each chosen clique $Q(S)$. Moreover, T

[3]Remember that x' is a copy of x and the latter is member of a clique.

is disjoint from $Q(T)$. But then we have done because with $|T \cap Q(S)| \leq 1$ for all $S \in \mathbf{S}$, and $|T \cap Q(T)| = 0$, we have $\alpha(H) \leq |\mathbf{S}| - 1$. $\square$

Inside the proof we talk about a 0,1-matrix, in particular, a clique-vertex incidence matrix. This can be seen as the first clue with the actual relation with linear programming, but we will not discuss about it in this survey.

3 Families of graphs that are perfect

Perfect graphs have come to be recognized as having a natural place in the world. Indeed, many well known families of graph are perfect.

3.1 Which subclasses?

The following it's a brief list of families of perfect graphs. There is much more out there, sure.

- Bipartite graphs
- The line graphs of bipartite graphs
- Interval graphs[4]
- Chordal graphs
- Split graphs[5]
- ... many more.

3.2 Trees and Chordal graphs

The very basic class of graphs that are obviously perfect is the class of trees.

Lemma 3. *Trees are perfect.*

Proof. It suffices to observe that, for any tree (forest) T, it holds $\chi(T) = \omega(T) = 2$. $\square$

Lemma 4. *Chordal graphs are perfect.*

Proof. (With the strong perfect graph theorem)
Let G be a chordal graph.

- There is no hole in G, by definition of a chordal graph.

[4]Vertices represent line intervals; and edges, their pairwise nonempty intersections.
[5]Graphs that can be partitioned into a clique and an independent set.

- Can there be an odd anti-hole? Since we do not have the property G *chordal* $\Leftrightarrow$ $\overline{G}$ *chordal* (see the "square"), we can't use the complement. Suppose G contains an odd anti-hole.
  - Either it is of size 5. Then C_5 is an induced subgraph of $\overline{G}$. But $\overline{C_5} = C_5$. So C_5 is an induced subgraph of G. Contradiction.
  - Or it is of size $5 + 2k$. Then P_5 is an induced subgraph of $\overline{G}$. But $\overline{P_5}$ is not chordal. Contradiction.

□

Proof. (Without the strong perfect graph theorem)
Since every induced subgraph of a chordal graph is chordal too, we need only to check that $\forall G$ *chordal,* $\chi(G) = \omega(G)$.
We know that $\chi(G) \geq \omega(G)$ (a clique needs a color per vertex)
Let G be a chordal graph. Let $(a_1, ..., a_n)$ be an elimination scheme. If we color each a_i (from a_n to a_1), according to the colors of its neighbors already colored, those form a clique with a_i so there is always a color suitable for a_i that is $\leq \omega(n)$. So G is $\omega(G)$-colourable. This means $\chi(G) \leq \omega(G)$.
We then have $\chi(G) = \omega(G)$. □

3.3 Computing the chromatic number

Let's see how to calculate the chromatic number for trees and chordal graphs and how much it costs.

Lemma 5. *Finding the chromatic number of a tree can be done in constant time.*

Proof. If it has exactly one vertex, its chromatic number is 1. Otherwise it is 2. □

Lemma 6. *Finding the chromatic number of a chordal graph can be done in quadratic time.*

Proof. Let G be a chordal graph. We use Lex-BFS to find $(a_1, ..., a_n)$, an elimination scheme (this is linear in the number of edges). If we color each a_i (from a_n to a_1), according to the colors of its neighbors already colored, those form a clique with a_i, we pick the smallest available color for a_i (the smallest that is not taken by the already colored neighbors). We keep track of the largest color used. When a_1 is colored, the largest color used corresponds to the chromatic number of the graph. This computes the chromatic number in quadratic time in the number of edges (which is actually not optimal, there exists a linear algorithm). Indeed, since every time we use a new color, it means that all the already-colored neighbors (which form a clique) used all the others. So we have a clique of size the new color. □

3.4 Split graphs

Given an undirected graph $G = (V, E)$, this is a *split graph* if there is a partition of its vertices such that the first subset of vertices is a stable set and the second one is a clique. Namely, $V = S + K$. An example of split graph is in figure 5.
There isn't any restriction on edges between vertices of S and vertices of K. In general, there is no one unique partition, neither will S (resp. K) necessarily be a maximal stable set (resp. clique).
Since a stable set in G is a clique in $\bar{G}$, and vice versa, we obtain a neat result.

Theorem 2. *An undirected graph G is a split graph if and only if its complement $\bar{G}$ is a split graph.*

The next theorem brings us many interesting information of split graphs.

Theorem 3. *Let G be a split graph whose vertices have been partitioned into a stable set S and a clique K. Exactly one of the following conditions holds:*

1. $|S| = \alpha(G)$ *and* $|K| = \omega(G)$ *(in this case the partition $S + K$ is unique)*
2. $|S| = \alpha(G)$ *and* $|K| = \omega(G) - 1$ *(in this case there exist an x in S such that $K + x$ is complete)*
3. $|S| = \alpha(G) - 1$ *and* $|K| = \omega(G)$ *(in this case there exist an y in K such that $S + y$ is stable)*

Proof. Admitted. For a detailed proof look at [MCG]. □

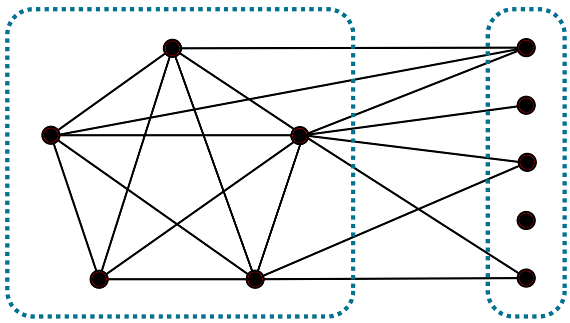

Figure 5: A split graph G

The next theorem is the main of this section. From it, we will conclude that split graphs are perfect.

Theorem 4. *Let G be an undirected graph. The following conditions are equivalent:*

1. *G is a split graph*

2. *G and $\bar{G}$ are chordal graphs*

3. *G contains no induced subgraph isomorphic to $2K_2$, C_4, or C_5.*

Proof. **1. implies 2.** Let $G = (V, E)$ have a vertex partition $V = S + K$ with S stable set and K clique. Now, suppose that G contains a chordless cycle C of length ≥ 4. At least one, at most two adjacent vertices of C would be in K.
Both cases would imply that S contains a pair of adjacent vertices, a contradiction. Therefore, G must be chordal. By the theorem 2, $\bar{G}$ is split, so $\bar{G}$ is chordal.
2. implies 3. Trivial.
3. implies 1. Let K be a maximum clique of G, chosen among all maximum cliques, so that $G[V - K]$ has fewest possible edges. We must show that $S = V - K$ is stable.
Suppose, on the contrary, that $G[S]$ has an edge xy. By the maximality of K, no vertex of S could be adjacent to every vertex of K. Moreover, if both x and y are adjacent to every vertex of K with the exception of the same single vertex z, then $K - \{z\} + \{x\} + \{y\}$ would be a complete set larger than K. Thus, there must exist distinct vertices $u, v \in K$ such that $xu \notin E$ and $yv \notin E$.
Since G contains neither an induced copy of $2K_2$ nor C_4, it follows that exactly one of the edges xv or yu is in G. Assume, w.l.g., that $xv \notin E$ and $yu \in E$. For any $w \in K - \{u, v\}$, if $yw \notin E$ and $xw \notin E$, then $G[\{x, y, u, w\}] \cong 2K_2$, whereas if $yw \notin E$ and $xw \in E$, then $G[\{x, y, u, w\}] \cong C_4$. Thus, y is adjacent to every vertex of $K - \{v\}$, and $K' = K - \{v\} + \{y\}$ is a maximal clique.
Since $G[V - K']$ can't have fewer edges than $G[v - K]$ has, it follows from the fact x is adjacent to y but not to v that there exist a vertex $t \neq y$ in $V - K$ which is adjacent to v but not to y. Now tx must be and edge of G, for otherwise $\{t, x, y, v\}$ would induce a copy of $2K_2$. Similarly,$tu \notin E$, for otherwise $\{t, x, y, u\}$ would induce a copy of C_4. However, this implies that $\{t, x, y, u, v\}$ induces a copy of C_5, a contradiction. Therefore, $S = V - K$ is stable, and G is a split graph. □

But then, if a split graph G is also chordal then it holds

Proposition 3. A split graph G is perfect.

Proof. From theorem 4, G is chordal. Because chordal graphs are perfect, so is G. □

4 The Strong Perfect Graph Theorem

Paul Seymour and his work associates have proved in 2002 the second conjecture made by Berge. The conjecture has stood for 40 years. Berge has died in 2002, but he was alive when the conjecture was proved. For further information, look at [Sey].

Theorem 5. *(The Strong Perfect Graph Theorem)*
A graph G is perfect if and only if G is Berge, that is, it contains no odd hole or antihole[6].

4.1 On trees, chordal graphs and split graphs

Let's see how to apply the latter theorem on the studied families of graphs. Since trees have no cycles, they are trivially *Berge*.
By definition, chordal graphs have no cycles of length ≥ 4. Then they are *Berge*. And finally, split graphs are also chordal graphs, and so they are *Berge* too.

References

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[Sey] "How the proof of the strong perfect graph conjecture was found", P. Seymour, 2006.

[6] A *hole* means an induced subgraph which is a cycle of length at least four, and an *antihole* is the complement.