International Journal of Foundations of Computer Science

A NOTE ON SYNCHRONIZED AUTOMATA AND ROAD COLORING PROBLEM[*] [†]

KAREL CULIK II
Department of Computer Science, University of South Carolina
Columbia, South Carolina 29208, USA.

JUHANI KARHUMÄKI
Department of Mathematics and Turku Centre for Computer Science, University of Turku
FIN-20014 Turku, Finland

and

JARKKO KARI[‡]
Department of Computer Science, 14 MacLean Hall, University of Iowa
Iowa City, Iowa 52242, USA

Received (received date)
Revised (revised date)
Communicated by Editor's name

ABSTRACT

We consider the problem of labeling a directed multigraph so that it becomes a synchronized finite automaton, as an ultimate goal to solve the famous Road Coloring Conjecture, cf. [1, 2]. We introduce a relabeling method which can be used for a large class of automata to improve their "degree of synchronization". This allows, for example, to formulate the conjecture in several equivalent ways.

Keywords: Finite automata, synchronization, road coloring problem

1. Introduction

Synchronization properties in automata theory are fundamental, and often at the same time very challenging. Two examples of such problems are as follows. Let us call a finite automaton $\mathcal{A}$ *synchronized* if there exists a word w which takes each state of $\mathcal{A}$ to a single special state s. Such a w is called *synchronizing* word for $\mathcal{A}$.

So-called *Cerny's Conjecture* [3, 12] claims that each synchronized automaton possesses a synchronizing word of length at most $(n-1)^2$, where n is the cardinality

[*] A preliminary version of the work was presented in the conference *Developments in language theory 2001.*

[†] Research supported by the Academy of Finland under grant 44087 and by NSF under grant CCR 97-33101.

[‡] Current address: Department of Mathematics, University of Turku, FIN-20014 Turku, Finland

of the state set of $\mathcal{A}$. Despite many attempts the conjecture in the general case is still unsolved, the best upper bound being cubic in n, cf. [8]. However, recently in [6] a natural extension of Cerny's Conjecture stated in [13] was shown to be false.

Cerny's Conjecture asks something about synchronized automata. *Road Coloring Problem*, in turn, asks for a dual task: change, if possible, an automaton to a synchronized one. More precisely, given a deterministic complete and strongly connected automaton, can it be relabeled to a synchronized automaton.

It is well known, cf. Lemma 2, that the Road Coloring Problem has a negative answer if the greatest common divisor of the lengths of all loops in $\mathcal{A}$ is larger than 1. In the opposite case - which due to the strong connectivity is equivalent to the existence of two loops of coprime lengths - the answer is conjectured to be affirmative. This is the *Road Coloring Conjecture, RC-conjecture* for short. In terms of graphs it is formulated as follows: Let us call a directed graph G *acceptable* if it is of uniform outdegree and strongly connected (i.e. for any pair (p, q) of vertices there is a path from p to q) and *primitive* if the greatest common divisor of lengths of its loops is one. The conjecture claims that each acceptable primitive graph can be labeled to a synchronized finite automaton.

Intuitively the above means that if a traveler in the network of colored roads modeled by an acceptable primitive graph gets lost, he can find a way back home by following a single instruction, the synchronizing word.

The Road Coloring Conjecture has attracted a lot of attention over the past 20 years. However, it has been established only in a very limited cases, cf. [4, 10, 11], and it is stated as a "notorious open problem" in [9].

We attempt to solve the problem by analyzing properties of different labelings of finite automata. In particular, we describe a method to increase the synchronization degree of an automaton by relabeling it in a suitable way. Here the synchronization degree of an automaton $\mathcal{A}$ is the minimal number $n_{\mathcal{A}}$ of states of $\mathcal{A}$ such that there exists a word w taking each state of $\mathcal{A}$ to one of these $n_{\mathcal{A}}$ states. Unfortunately, our method does not work for all labelings, but it does work for a quite large subclass of labelings, and, moreover, allows to formulate the RC-conjecture in two equivalent ways.

This paper is organized as follows. In Section 2 we fix our terminology and formulate several conjectures connected to synchronization properties including the Road Coloring Conjecture. In Section 3 we introduce an automaton which, for a given automaton $\mathcal{A}$, computes all words having the maximal synchronizing effect, in particular all synchronizing words, if the automaton is synchronized. In Section 4 we relate the synchronization to certain equivalence relations. Finally, Section 5 introduces our relabeling technique to improve the synchronization.

2. Preliminaries

In this section we fix the necessary terminology, cf. [5], and formulate several conjectures including the Road Coloring Conjecture.

Let $G = (V, E)$ be a directed graph with vertex set V and edge set E, where multiple edges are allowed. We consider only graphs which are

(i) *strongly connected*, and

(ii) of *uniform outdegree*, i.e. all vertices have the same outdegree, say n.

Such a graph is called *acceptable*. Clearly, each acceptable graph can be labeled by an n-letter alphabet to yield a deterministic strongly connected and complete automaton without initial and final states. By a *labeling* of an acceptable graph we mean such a labeling, or also an automaton defined by such a labeling.

Let G be an acceptable graph and L_G the set of its all loops. We call G *primitive* if the greatest common divisor of the lengths of the loops in L_G is equal to 1; otherwise G is *imprimitive*. Further we call G *cyclic* if there exist an $N \geq 2$ and a partition of the vertex set V of G into the classes $V_0, \ldots, V_{N-1}$ such that whenever $p \longrightarrow q$ is an edge in G, then $p \in V_i$ and $q \in V_{i+1 \pmod N}$ for some $i = 0, \ldots, N-1$. Otherwise G is *acyclic*. We have an easy connection:

Lemma 1 *An acceptable graph G is imprimitive iff it is cyclic.*

Proof. Trivially every cyclic graph is imprimitive: the cycle length N divides the lengths of all loops in G. Consider the other direction. Let G be imprimitive, and N the gcd of the lengths of all loops of G. We color the vertex set V of G by the function $c : V \rightarrow \{0, \ldots, N-1\}$ as follows: Let T be a spanning tree of G rooted at r. We set $c(r) = 0$ and $c(t) = c(s) + 1$, whenever T contains an edge $s \rightarrow t$.

We claim that for all edges $e : p \rightarrow q$ in G we have $c(q) \equiv c(p) + 1 \pmod N$. For edges in T that is clear. For other edges we denote the path from r to p in T by $\alpha_0\alpha_p$ and the path from r to q in T by $\alpha_0\alpha_q$, where α_0 is the maximal common prefix of these paths. Moreover, let α_r be a path from q to r in G. Then G contains the cycles $\alpha_p e \alpha_r \alpha_0$ and $\alpha_q \alpha_r \alpha_0$. Since the lengths of both of these are a multiple of N we have $|\alpha_q| \equiv |\alpha_p| + 1 \pmod N$. This proves our claim.

□

In order to formulate our conjectures we recall some terminology of automata. Let $\mathcal{A} = (Q, \Sigma, \delta)$ be a complete deterministic finite automaton without final and initial states. We say that the automaton $\mathcal{A}$ is *synchronized* at state $s \in Q$ if there exists a word $w \in \Sigma^*$ that takes each state q of Q into s, i.e. $\delta(q, w) = s$ for all $q \in Q$. The word w is called a *synchronizing* word for $\mathcal{A}$. Clearly, if a strongly connected automaton is synchronized at a given state it is so at any of its states.

Now, we extend the above to graphs. We say that an acceptable graph is *synchronizable* if it has a labeling making it a synchronized automaton. Note that originally in [1] the word "collapsible " was used instead of "synchronizable". Now, the conjecture can be stated as follows:

Road Coloring Conjecture. Each primitive acceptable graph is synchronizable.

The conjecture, if true, is optimal:

Lemma 2 *Each imprimitive graph is not synchronizable.*

Proof. Clearly a cyclic graph is not synchronizable, so Lemma 2 follows from Lemma 1. □

Next we define a few properties related to labelings of an acceptable primitive graph. Let δ be such a labeling. We say that a pair p, q of vertices is *reducible*, in symbols $p \sim q$, if there exists a word w such that $\delta(p, w) = \delta(q, w)$, i.e. word w takes p and q to the same state. Accordingly such a δ is called (p, q)-*synchronized*. We say that a vertex pair p, q is *stable*, denoted $p \equiv q$, if for every word u there exists a word w such that $\delta(p, uw) = \delta(q, uw)$. We also say that δ is (p, q)-*stable*.

Clearly, the reducibility and the stability relations $\sim$ and $\equiv$ are symmetric and reflexive. The reducibility relation is not always transitive (see Example 2). In contrast, the stability relation is transitive:

Lemma 3 *The stability relation $\equiv$ is an equivalence relation.*

Proof. Let us prove transitivity. Let $p \equiv q$ and $q \equiv r$, and let u be an arbitrary word. To prove $p \equiv r$ we need to show that there exists a word w such that $\delta(p, uw) = \delta(r, uw)$. Because $p \equiv q$ there exists a word w_1 such that $\delta(p, uw_1) = \delta(q, uw_1)$, and because $q \equiv r$ there exists a word w_2 such that $\delta(q, uw_1w_2) = \delta(r, uw_1w_2)$. Then $\delta(p, uw_1w_2) = \delta(q, uw_1w_2) = \delta(r, uw_1w_2)$, so we can choose $w = w_1w_2$. □

We call the equivalence classes of the stability relation the *stability classes* of the automaton. By the definition of stability, the transition function δ is consistent with respect to the stability relation $\equiv$, that is, the relation $\equiv$ is a *congruence*.

If also the reducibility relation happens to be an equivalence relation, and nondiscrete, then the labeling δ is called *strong*. (Recall that an equivalence relation is called discrete if each element is in the relation with itself only. A nondiscrete equivalence relates at least one pair of distinct elements.) The nondiscreteness is to avoid some trivial exceptions in our later considerations. We show in Section 4 that if δ is strong then the reducibility and stability relations coincide.

Now we formulate several conjectures. The first one is a weaker version of the Road Coloring Conjecture.

Conjecture A. Let G be acceptable primitive graph. For each pair (p, q) of vertices of G there exists a (p, q)-synchronized labeling.

Conjecture A seems to be much weaker than the RC-conjecture but it might be equivalent to it. The two other conjectures we formulate are, as we shall show, equivalent to the Road Coloring Conjecture.

Conjecture B. For each acceptable primitive graph (with more than one vertex) there exists a strong labeling.

Conjecture C. For each acceptable primitive graph (with more than one vertex) there exists a labeling such that the stability relation is non-discrete.

In all examples we consider only binary alphabets Σ. Indeed, this case seems to capture the difficulty of the problem. The following example illustrates our conjectures

Example 1. Consider the automata shown in Figure 1. The automaton $\mathcal{A}$ possesses a synchronizing word, for example *baaab*, while the automaton $\mathcal{B}$ does not possess any. The reducibility relations are the full relation and the relation $\{(2,3),(3,2),(1,4),(4,1)\}$, respectively. Consequently also in the latter case the reducibility is an equivalence relation, and hence the labeling is strong. In both automata all reducible pairs are also stable, so relations $\sim$ and $\equiv$ are identical.

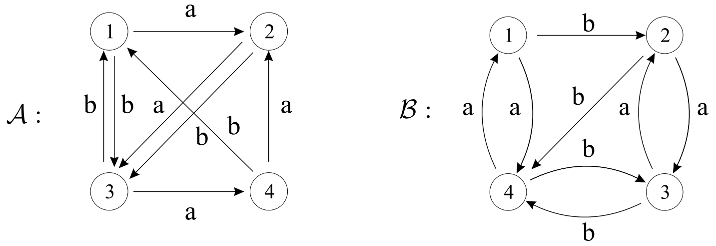

Fig. 1. Automata $\mathcal{A}$ and $\mathcal{B}$

3. An automaton for synchronizing words

Let G be an acceptable graph and $\mathcal{A}$ the automaton obtained from it via the labeling δ. Hence $\mathcal{A} = (Q, \Sigma, \delta)$ is a complete deterministic automaton without initial and final states. We define another automaton $\mathcal{A}_s$ as follows:

$$\mathcal{A}_s = (2^Q, \Sigma, \delta_s, Q),$$

where Q is the initial state and the transition function δ_s is defined by

$$\delta_s(P, q) = \bigcup_{p \in P} \delta(p, a) \quad \text{for } P \subseteq Q,\, a \in \Sigma. \tag{1}$$

Clearly $\mathcal{A}_s$ is complete and deterministic, and, moreover, a word w is synchronizing for $\mathcal{A}$ if and only if $\delta_s(Q, w)$ is singleton. Hence, we have

Lemma 4 *The set of all synchronizing words for $\mathcal{A}$ is computed by the automaton $(2^Q, \Sigma, \delta_s, Q, F)$, where the set F of final states consists of all singletons of the power set 2^Q.* □

We say that the *synchronization degree* of $\mathcal{A}$ is

$$\begin{aligned} n_{\mathcal{A}} &= \min_{w \in \Sigma^*} \{\mathrm{card}(P) \mid \delta(Q, w) = P\} \\ &= \min\{\mathrm{card}(P) \mid P \text{ is reachable state of } \mathcal{A}_s\}. \end{aligned}$$

Consequently, $\mathcal{A}$ is synchronized iff $n_{\mathcal{A}}$ is equal to 1. Let us define

$$Q_{\min} = \{P \mid \mathrm{card}(P) = n_{\mathcal{A}} \text{ and } P \text{ is reachable in } \mathcal{A}_s\}.$$

Using these notions we define another automaton $\mathcal{A}_{\min}$ as follows:

$$\mathcal{A}_{\min} = (Q_{\min}, \Sigma, \delta_s).$$

The automaton $\mathcal{A}_{\min}$ plays an important role in our subsequent considerations.

Lemma 5 *The automaton $\mathcal{A}_{\min}$ is deterministic, complete and strongly connected.*

Proof. First, by (1) and the minimality of $n_{\mathcal{A}}$, the automaton $\mathcal{A}_{\min}$ is well defined, deterministic and complete.

Second, to prove that $\mathcal{A}_{\min}$ is strongly connected let P_1 and P_2 be states of $\mathcal{A}_{\min}$. This means that there exists words w_1 and w_2 such that $\delta_s(Q, w_i) = P_i$ for $i = 1, 2$. Since $P_1 \subseteq Q$, necessarily $\delta_s(P_1, w_2) \subseteq \delta_s(Q, w_2) = P_2$. Hence, by the minimality of $n_{\mathcal{A}}$, $\delta_s(P_1, w_2) = P_2$. Similarly, $\delta_s(P_2, w_1) = P_1$, proving the strong connectivity. □

To illustrate the above notions we return to our example.

Example 1. *(continued).* Consider automata $\mathcal{A}$ and $\mathcal{B}$ of Example 1. The automata $\mathcal{A}_s$, $\mathcal{A}_{\min}$, $\mathcal{B}_s$ and $\mathcal{B}_{\min}$ are shown in Figure 2, where the min-automata are shown by the dash lines. It follows that $\mathcal{A}$ is synchronized and its shortest synchronizing word is *baaab*. On the other hand, $\mathcal{B}$ is not synchronized - its synchronization degree is 2.

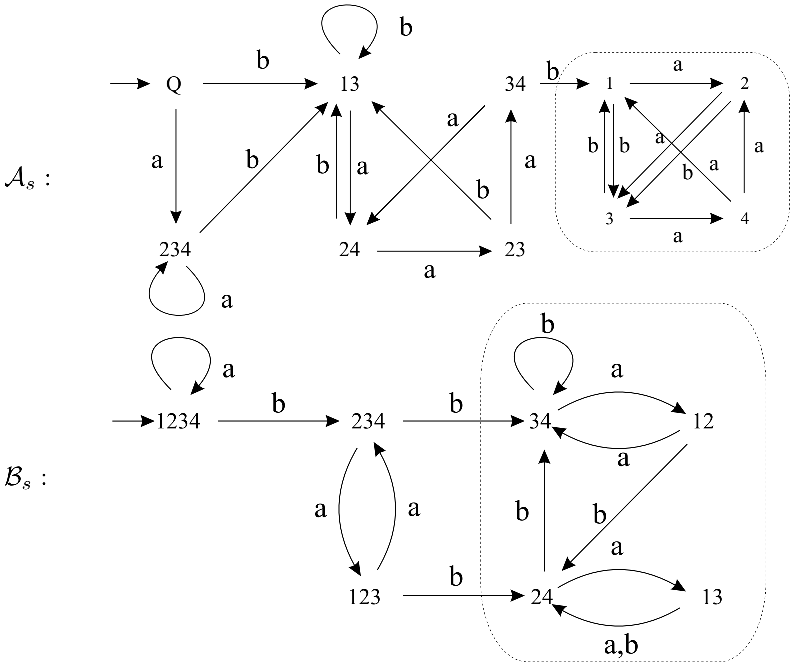

Fig. 2. Automata $\mathcal{A}_s$, $\mathcal{B}_s$, $\mathcal{A}_{\min}$, and $\mathcal{B}_{\min}$, the latter ones being those shown by dashlines.

Let us analyze a bit more the above automata $\mathcal{A}_s$ and $\mathcal{A}_{\min}$. Let w be a word leading the initial state Q of $\mathcal{A}_s$ into $Q_{\min}$, in other words

$$\delta_s(Q, w) = \{p_1, \ldots, p_{n_{\mathcal{A}}}\}. \tag{2}$$

Then w defines an equivalence relation $\sim_w$ on Q, where the equivalence classes consist of subsets $\{q \mid \delta(q, w) = p_i\}$ for $i = 1, \ldots, n_{\mathcal{A}}$. Of course, in general, such partitions need not be unique. However, the uniqueness is characterized by the strongness of the labeling, cf. Theorem 1 in the next section.

4. Equivalence relations vs. synchronization

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton obtained from an acceptable primitive graph via a labeling δ. From the synchronization point of view fundamental notions are the reducibility and stability relations $\sim$ and $\equiv$. As we already noted the reducibility relation $\sim$ is reflexive and symmetric, but not necessarily transitive, as shown by the next example.

Example 2. Consider an automaton $\mathcal{C}$ and its variants $\mathcal{C}_s$ and $\mathcal{C}_{\min}$ shown in Figure 3. In this case states 1 and 2 are reducible by b, and states 1 and 3 by ab, but 2 and 3 are not reducible by any word. Hence, the relation $\sim$ is not an equivalence relation. This is connected to the fact that the partitions of Q defined by different words (as was explained in Section 3) need not coincide. Indeed, the word b defines the partition $\{\{1,2\},\{3,4\}\}$, while the word ab defines the partition $\{\{1,3\},\{2,4\}\}$. There are no non-trivial stable pairs of states so the stability relation $\equiv$ of $\mathcal{C}$ is discrete.

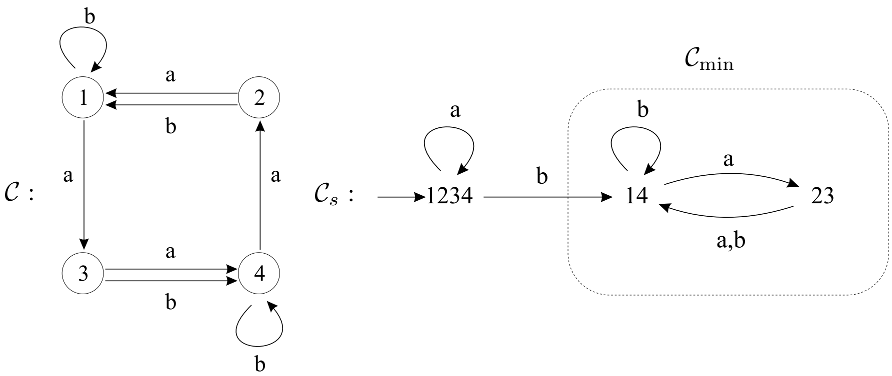

Fig. 3. Automata $\mathcal{C}$ and $\mathcal{C}_s$ as well as $\mathcal{C}_{\min}$ obtained from the latter.

We have the following important characterization.

Theorem 1 *The labeling δ is strong if and only if the partitions $\sim_w$ by (2) in Section 3 are independent of w. If δ is strong then the reducibility relation $\sim$ and the stability relation $\equiv$ coincide.*

Proof. Assume first that the labeling δ is strong, i.e., the reducibility relation $\sim$ is an equivalence relation. Since the states of the automaton $\mathcal{A}_{\min}$ are pairwise irreducible, the number of equivalence classes under $\sim$ is $n_{\mathcal{A}}$. On the other hand, by the construction of $\mathcal{A}_{\min}$, any partition in (2) induced by a fixed word w is of the same cardinality. So the independence of these partitions follows since any class in these partitions is a subset of an equivalence class of the reducibility relation.

Conversely, if the partitions in (2) are independent of w, then clearly any two states that are reducible belong to a same equivalence class in (2). Hence, the reducibility matches with the unique relation, and consequently is an equivalence relation.

The second sentence follows from the fact that if w satisfies (2) then for every word u also uw satisfies (2). In general it is possible that $\sim_w$ and $\sim_{uw}$ are two different relations, but if δ is strong then it follows from the first part of the theorem that $\sim_w$ and $\sim_{uw}$ are identical. If $p \sim q$ then $p \sim_w q$ for some w satisfying (2), which means that for every word u we have $p \sim_{uw} q$, i.e., $p \equiv q$. □

Automata like in Example 2 are problematic for our general approach. Indeed, as we shall see, whenever the labeling is strong we can improve the synchronization degree of a nonsynchronized automaton.

5. Improving the synchronization

In this section we introduce our method to relabel automata. Let $\mathcal{P}$ be an arbitrary partition of the state set Q, and let δ' be a relabeling of the automata. We say that the relabeling respects partition $\mathcal{P}$ (and the corresponding equivalence relation of Q) iff for each equivalence class $P \in \mathcal{P}$ there exists a permutation π of the input alphabet such that for every $p \in P$ and $a \in \Sigma$ we have

$$\delta'(p, a) = \delta(p, \pi(a)).$$

In the case of a binary alphabet this means that for each equivalence class P either all labels of transitions from P are changed or none of them is changed.

The reason to introduce this notion is the following: If in the original automaton we have computations

$$\begin{array}{ll} \delta: & p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} p_n \\ & \text{and} \\ & q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} q_n \end{array}$$

where for every i states p_i and q_i are in the same equivalence class of $\mathcal{P}$, and if δ' is a relabeling that respects $\mathcal{P}$, then in the relabeled automaton we have computations

$$\begin{array}{ll} \delta': & p_0 \xrightarrow{a'_1} p_1 \xrightarrow{a'_2} p_2 \xrightarrow{a'_3} \cdots \xrightarrow{a'_n} p_n \\ & \text{and} \\ & q_0 \xrightarrow{a'_1} q_1 \xrightarrow{a'_2} q_2 \xrightarrow{a'_3} \cdots \xrightarrow{a'_n} q_n. \end{array}$$

Each letter a'_i is obtained from the corresponding letter a_i using the relabeling that was applied at the equivalence class containing p_{i-1} and q_{i-1}. In particular for $p_n = q_n$ we get the result that our relabelings do not invalidate $\mathcal{P}$-consistent synchronizing computations.

Theorem 2 *Road coloring conjecture is equivalent to Conjecture C.*

Proof. Clearly the RC-conjecture implies conjecture C. Indeed, the stability relation of a synchronized automaton is the full relation. Assume then that Conjecture C is true. We use mathematical induction on the number of vertices of G to prove that every primitive acceptable G has a synchronized labeling. Let v be the number of vertices and assume that the RC conjecture is true for all graphs with fewer vertices.

It follows from Conjecture C that there exists a labeling of G into automaton A such that the stability relation is non-discrete. Because the stability relation is a congruence we have a factor automaton $F = (A/\equiv)$ whose states are the stability classes, and transitions are inherited from A. Any labeling of F induces a relabeling A' of A that respects the stability classes. This means that if p and q are stable in A they are stable in A' as well.

The factor automaton F is primitive. Indeed, if number n divides the lengths of all loops in F then n also divides the lengths of all loops in A. The factor automaton is also acceptable: it is strongly connected because A is.

Because the factor automaton F has fewer states than A we can use the inductive hypothesis and conclude that there exists a synchronized relabeling F' of F. Let us show that the induced relabeling of A into A' is also synchronized. Let p and q be two arbitrary vertices of G, and let p_F and q_F be the corresponding states in F, that is, the stability classes of p and q in automaton A. Because F' is synchronized there exists a word w such that in automaton F' we have $\delta_{F'}(p_F, w) = \delta_{F'}(q_F, w)$. This means that in automaton A' states $\delta_{A'}(p, w)$ and $\delta_{A'}(q, w)$ are stable, hence reducible. □

Corollary 1 *Road coloring conjecture is equivalent to Conjecture B.*

Proof. Clearly the road coloring conjecture implies conjecture B, and Conjecture B implies Conjecture C because every strong labeling has a non-discrete stability relation (Theorem 1). According to Theorem 2 Conjecture C implies the road coloring conjecture, so all three conjectures are equivalent. □

Before continuing let us return to Example 1.

Example 1. *(continued)* For the automaton $\mathcal{B}$ the min-automaton $\mathcal{B}_{\min}$ and the factor automaton $\mathcal{B}_F$ are as shown in Figure 4. To synchronize the partition automaton $\mathcal{B}_F$ we switch a and b in the labels starting from the state 23. Then the corresponding automata $\mathcal{B}'$ and $\mathcal{B}'_s$ are as shown in Figure 5. Hence, $\mathcal{B}'$ indeed is synchronized.

Our next result is unconditional.

Theorem 3 *For any acceptable primitive graph G there exists a labeling such that the transitive closure of the reducibility relation $\sim$ is the full relation.*

Proof. Let $\sim^*$ denote the transitive closure of the reducibility relation $\sim$. We use our relabeling technique over the equivalence relation $\sim^*$.

Let G be an acceptable primitive graph, let δ be a labeling and let $\sim$ be the corresponding reducibility relation. If δ' is a relabeling that respects the transitive closure $\sim^*$ then any two states p and q that are reducible under δ are reducible under δ' as well. This follows from the discussion in the beginning of this section.

Assume that $\sim^*$ is not the full relation. Because G is strongly connected and not cyclic it must contain two edges $p \longrightarrow x$ and $q \longrightarrow y$ such that $x \sim^* y$ but $p \not\sim^* q$. Without loss of generality we may assume that $x \sim y$: In any case we have

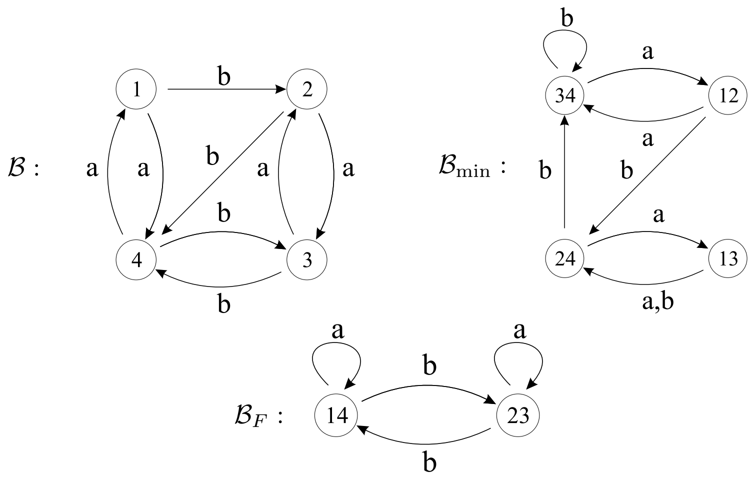

Fig. 4. Automaton $\mathcal{B}$, its min-automaton $\mathcal{B}_{\min}$, and its factor automaton $\mathcal{B}_F$.

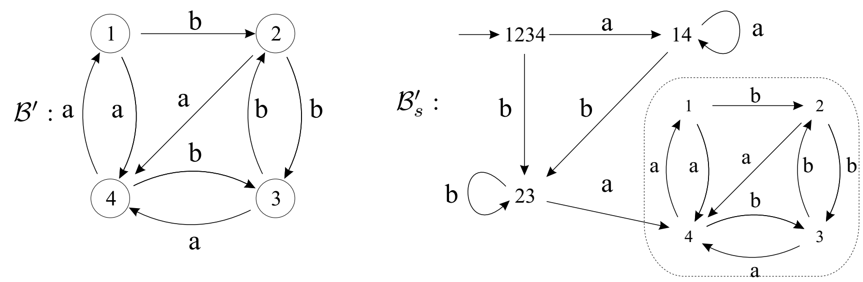

Fig. 5. Automaton $\mathcal{B}'$ obtained from $\mathcal{B}$ by relabeling and its variant $\mathcal{B}'_s$.

a sequence $x = x_1, x_2, \ldots, x_n = y$ of vertices such that $x_i \sim x_{i+1}$ for every i. For each i choose a vertex p_i such that $p_i \longrightarrow x_i$ is an edge, and $p_1 = p$ and $p_n = q$. Because p_1 and p_n are not in relation $\sim^*$ there must exist i such that p_i and p_{i+i} are not in relation $\sim^*$. Then we may choose x_i as x, x_{i+1} as y, p_i as p and p_{i+i} as q.

Let δ' be a relabeling that respects $\sim^*$ such that edges $p \longrightarrow x$ and $q \longrightarrow y$ have the same label a. Only the labels on the equivalence class containing p need to be changed. States p and q are reducible under the new labeling δ'. Indeed, letter a maps them into reducible states x and y. As the relabeling respects $\sim^*$ anything reducible under δ is reducible under δ'. We conclude that our relabeling decreased the index of the equivalence relation $\sim^*$.

Using mathematical induction on the index of $\sim^*$ we get the desired result. □

As a corollary of the theorem one can prove that any acceptable primitive graph G is *commutatively synchronizable*. Two words are called *commutatively equivalent* if one is obtained from the other by reordering its letters. We call an automaton *commutatively synchronized* if there exists a word w such that, for each vertex q, there exists a word w_q commutatively equivalent to w such that $\delta(p, w_p) = \delta(q, w_q)$ for all p, $q \in Q$. As in the non-commutative case, it is enough to commutatively synchronize $n-1$ pairs of states, one after the other. So an automaton is commutatively synchronized iff for every two states p and q there exist two commutatively equivalent words w_p and w_q such that $\delta(p, w_p) = \delta(q, w_q)$. If this is the case we say that pair p, q is *commutatively reducible*, and use the notation $p \sim_{com} q$.

Lemma 6 *The relation $\sim_{com}$ is an equivalence relation.*

Proof. We only need to prove transitivity. Let $p \sim_{com} q$ and $q \sim_{com} r$, and let w_p and w_q be commutatively equivalent words such that $\delta(p, w_p) = \delta(q, w_q) = s$ and let u_q and u_r be commutatively equivalent words such that $\delta(q, u_q) = \delta(r, u_r) = t$. Because G is strongly connected there exist words α and β such that $\delta(s, \alpha) = q = \delta(t, \beta)$. See Figure 6 for an illustration.

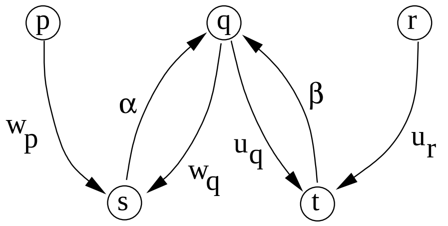

Fig. 6. Proof that commutative reducibility is transitive.

Because

$$\delta(p, w_p \alpha u_q \beta) = q = \delta(r, u_r \beta w_q \alpha)$$

and because words $w_p \alpha u_q \beta$ and $u_r \beta w_q \alpha$ are commutatively equivalent, we have $p \sim_{com} r$. □

Corollary 2 *Any acceptable primitive graph G is commutatively synchronizable.*

Proof. Take any labeling of G such that $\sim^*$ is the full relation (Theorem 3). Because $\sim \subseteq \sim_{com}$ and because $\sim_{com}$ is an equivalence relation (Lemma 6) we have $\sim^* \subseteq \sim_{com}$. So $\sim_{com}$ is the full relation and the labeling is commutatively synchronized. □

We do not claim that Theorem 3 and Corollary 2 are important from the point of view of synchronization, but they nicely emphasize the usefulness of our relabeling techniques.

Another application of the technique was recently invented in [7]. A directed graph is called *Eulerian* if every node has identical in- and outdegree. An acceptable Eulerian graph has therefore uniform in- and outdegrees.

The following simple idea guarantees a labeling with non-discrete stability relation for any acceptable and primitive Eulerian graph. We start with a labeling that is completely non-synchronized. In other words, we choose a labeling such that every letter specifies a permutation of the state set. An easy application of the marriage theorem shows that such a labeling must exist [7]. Then we swap the labels of two edges $r \longrightarrow s$ and $r \longrightarrow t$ where $s \neq t$. Such edges must exist if the graph is primitive and has at least two vertices. Let a and b be the labels of the two edges so that after the swap there are two edges with label a entering state s and two edges with label b entering state t. Let $\mathcal{A} = (Q, \Sigma, \delta)$ be the automaton after the swap. Let us show that $s \equiv t$.

First we rephrase an interesting result proved in 1990 by J.Friedman [4]. In the context of Eulerian graphs his result states that the equivalence classes of relations $\sim_w$ defined by (2) in Section 3 all have the same cardinality $\mathrm{card}(Q)/n_{\mathcal{A}}$. This implies that $s \sim_w t$. Namely, if there would exist an equivalence class S such that $s \in S$ but $t \notin S$ then the set

$$S' = \{p \mid \delta(p, a) \in S\}$$

has greater cardinality than S simply because s has two incoming edges with label a and all other states of S have one incoming a. But the set S' is synchronized by word aw so according to the Friedman's result it should have the same cardinality as S.

We conclude that s and t are synchronized by any word w that leads from Q into $Q_{\min}$ in $\mathcal{A}_s$. If w is such a word then also uw is such a word, for every word u. This implies that $s \equiv t$, and the stability relation $\equiv$ is non-discrete.

It is fairly easy to see that the factor automaton $F = (A/\equiv)$ is acceptable, primitive and Eulerian and therefore the relabaling technique can be used to find a synchronized labeling. We have [7]

Theorem 4 *The Road Coloring Conjecture is true for graphs with uniform in- and outdegrees.* □

References

1. R. L. Adler, L. W. Goodwyn and B. Weiss, *Equivalence of Topological Markov Shifts*, Israel J. Math. 27, 49-63, (1977).

2. M.-P. Beal and D. Perrin, *Symbolic Dynamics and Finite automata*, in: G. Rozenberg and A. Salomaa (eds.) Handbook of Formal Languages Vol II, 49-63, (1997).

3. J. Černý, *Poznámka k homogennym experimenton s konečnými automatmi*, Mat. fyz. čas. SAV 14, 208-215, (1964).

4. J. Friedman *On the road coloring problem*, Math. Soc. 110, 1133-1135, (1990).

5. J. E. Hopcroft and J. D. Ullman, *Introduction to Automata Theory, Languages and Computation*, (Addison-Wesley 1979).

6. J. Kari, *A counter example to a conjecture concerning synchronizing words in finite automata*, EATCS Bull. 73, 146, (2001).

7. J. Kari, *Synchronizing finite automata on Eulerian digraphs*, Proceedings of Mathematical Foundations of Computer Science 2001, LNCS 2136, 432-438, (2001).

8. A. Kljachko, I. Rystsow and K. Spivak, *Extended combinatorial problem concerning the length of the reset word in a finite automaton*, Cybernetics 23, 165-170, (1987).

9. D. Lind and B. Marcus, *An Introduction of Symbolic Dynamics and Coding*, (Cambridge Univ. Press, 1995).

10. G. L. O'Brien, *The road coloring problem*, Israel J. Math. 39, 145-154, (1981).

11. D. Perrin and M.P. Schüzenberger, *Synchronizing prefix codes and automata and the road coloring problem*, in: P. Walters (ed.) Symbolic Dynamics and its Applications, Contemporary Mathematics 135, 295-318, (1991).

12. J.-E. Pin, *Le probleme de la conjecture de Černý*, These de 3^e cycle, Universite de Paris VI (1978).

13. J.-E. Pin, *On two combinatorial problems arising from automata theory*, Annals of Discrete Mathematics 17, 535–548, (1983).