Performance Evaluation in Networks

Discrete time
Markov Chains (MC)
Discrete time Markov Chain (MC) : definition

Process \((X_n)_{n \in \mathbb{N}}\) where \(X_n\) r.v. over \((\Omega, \mathcal{F}, \mathbb{P})\), with values in \(E\).

Definition (Markov Chain : MC)

\((X_n)\) markovian if \(\forall n \in \mathbb{N}, \forall x_0, \ldots, x_n, x_{n+1} \in E\) (space of states),

\[
\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n)
\]

subject to \(\mathbb{P}(X_n = x_n, \ldots, X_0 = x_0) \neq 0\).
Discrete time Markov Chain (MC) : definition

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\[ P(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} | X_n = x_n) \]

subject to \(P(X_n = x_n, \ldots, X_0 = x_0) \neq 0\).

⚠️ **Stochastic process**

All r.v. \(X_n\) are defined over the *same* probabilistic spacee \((\Omega, \mathcal{F}, P)\) with values in the *same* space \(E \rightarrow \) each realization \(\omega \in \Omega\) yields a trajectory \(X_0(\omega), X_1(\omega), X_2(\omega), \ldots\) within \(E\).
Discrete time Markov Chain (MC) : definition

Process \((X_n)_{n \in \mathbb{N}}\) where \(X_n\) r.v. over \((\Omega, \mathcal{F}, \mathbb{P})\), with values in \(E\).

**Definition (Markov Chain : MC)**

\((X_n)\) **markovian** if for all \(n \in \mathbb{N}\), \(x_0, \ldots, x_n, x_{n+1} \in E\) (space of states),

\[ P(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} | X_n = x_n) \]

subject to \(P(X_n = x_n, \ldots, X_0 = x_0) \neq 0\).

⚠️ **Convention for conditional probas**

All formulas from the course with conditional probas are valid only if well defined: \(P(A|B)\) well defined if \(P(B) \neq 0\).

With this convention, the note “subject to ...” will be omitted for now, but stay alert in practice.
Discrete time Markov Chain (MC) : definition

Process \((X_n)_{n \in \mathbb{N}}\) where \(X_n\) r.v. over \((\Omega, \mathcal{F}, \mathbb{P})\), with values in \(E\).

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\[ \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n) \]

subject to \(\mathbb{P}(X_n = x_n, \ldots, X_0 = x_0) \neq 0\).

**Intuition** : “future only depends on present”, “memoryless”, ...

**Definition (Time Homogeneous MC : HMC)**

\((X_n)\) homogeneous if \(\forall n \in \mathbb{N}, \forall i, j \in E, \mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i)\)

**Definition (Transition matrix & graph of Homogeneous MC)**

- **Transition matrix** : \(P = (p_{ij})_{i,j \in E}\) with \(p_{ij} = \mathbb{P}(X_1 = j | X_0 = i)\)
- **Transition graph** : vertices \(= E\), edge \((i, j)\) if \(p_{ij} > 0\) (weight \(p_{ij}\))
Discrete Time Markov Chain (MC) : examples?

Discrete time MC / First properties
Classification
Asymptotic behaviour
Tools for analysis

Definition
Chapman-Kolmogorov
Stopping time / Strong Markov property
“One step forward”

Proposition (Characteristic example)
Let \((U_n)_{n \in \mathbb{N}}\) i.i.d. sequence of r.v. with values in \(F\), finite or countable space, \(X_0\) r.v. with values in \(E\) and independent of the sequence \((U_n)_{n \in \mathbb{N}}\), then the recurrence equation
\[
X_{n+1} = f(X_n, U_{n+1})
\]
define an homogeneous MC with values in \(E\).
Discrete Time Markov Chain (MC) : examples

- Jeu de l’oie / Snakes and ladders
- Sequence of i.i.d. r.v. for any law over $E$.
- Uniform random walk over $\mathbb{N}^d$ or $\mathbb{Z}^d$.
- Some randomized algorithms, e.g. in system/network protocols.

**Proposition (Characteristic example)**

Let $(U_n)_{n \in \mathbb{N}^*}$ i.i.d. sequence of r.v. with values in $F$, $E$ finite or countable space, $f$ map $E \times F \to E$, $X_0$ r.v. with values in $E$ and independent of the sequence $(U_n)$, then the recurrence equation $X_{n+1} = f(X_n, U_{n+1})$ define an homogeneous MC with values in $E$. 
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Transition matrix & graph

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\[ P = \text{stochastic matrix} : \]
- positive coeff : \( \forall i, j, p_{ij} \geq 0 \)
- \( \sum \text{over line} = 1 \) : \( \forall i, \sum_j p_{ij} = 1 \)

MC = “random walk” :
realization \( X_0(\omega), X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega), X_5(\omega), \ldots \) : walk in the transition graph

Important Notation: let \( i \in E \), \( P_i(A) \overset{\text{def}}{=} P(A | X_0 = i) \) for event \( A \)
\( E_i(Z) \overset{\text{def}}{=} E(Z | X_0 = i) = \sum_z z \cdot P(Z = z | X_0 = i) \) for real r.v. \( Z \) (\( \sum \text{ou f} \))
### Transition matrix & graph

**Stochastic matrix $P$:**
- Positive coefficients: $\forall i, j, \ p_{ij} \geq 0$
- Sum over line = 1: $\forall i, \sum_j p_{ij} = 1$

**Random walk (MC):**
- Realization $X_0(\omega), X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega), X_5(\omega), ...$
- Walk in the transition graph

**Important Notation:**
- Let $i \in E$, $P_i(A) \overset{\text{def}}{=} P(A|X_0 = i)$ for event $A$
- $E_i(Z) \overset{\text{def}}{=} E(Z|X_0 = i) = \sum_z z \cdot P(Z = z|X_0 = i)$ for real r.v. $Z$ (Sum of $\Sigma$)
Transition matrix & graph

\[
P = \text{stochastic matrix:}
\]
- positive coeff: \( \forall i, j, p_{ij} \geq 0 \)
- \( \sum \) over line = 1: \( \forall i, \sum_j p_{ij} = 1 \)

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 1/2 & 1/4 & 0 & 1/4 & 0 \\
2 & 1/2 & 0 & 1/2 & 0 & 0 \\
3 & 0 & 4/5 & 0 & 0 & 1/5 \\
4 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 1/3 & 0 & 2/3 & 0 \\
\end{array}
\]

\[
\text{MC = \text{“random walk”:}}
\]
- realization \( X_0(\omega), X_1(\omega), X_2(\omega), \ldots \)
- walk in the transition graph

Important Notation: let \( i \in E, \mathbb{P}_i(A) \overset{\text{def}}{=} \mathbb{P}(A | X_0 = i) \) for event \( A \)
\[
\mathbb{E}_i(Z) \overset{\text{def}}{=} \mathbb{E}(Z | X_0 = i) = \sum_z z \cdot \mathbb{P}(Z = z | X_0 = i)
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for real r.v. \( Z \) (\( \sum \text{ouf} \))
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\[
P = \text{stochastic matrix}:
\begin{align*}
&\text{positive coeff}: \forall i,j, \ p_{ij} \geq 0 \\
&\text{over line } = 1: \forall i, \sum_j p_{ij} = 1
\end{align*}
\]

\[
P = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 1/2 & 1/4 & 0 & 1/4 & 0 \\
2 & 1/2 & 0 & 1/2 & 0 & 0 \\
3 & 0 & 4/5 & 0 & 0 & 1/5 \\
4 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 1/3 & 0 & 2/3 & 0
\end{pmatrix}
\]

\[
\begin{align*}
\text{MC} = \text{“random walk”} : \\
\text{realization } x_0(\omega), x_1(\omega), x_2(\omega), x_3(\omega), x_4(\omega), x_5(\omega), \ldots \\
\text{walk in the transition graph}
\end{align*}
\]

Important Notation: let \( i \in E \), \( \mathbb{P}_i(A) \overset{\text{def}}{=} \mathbb{P}(A|X_0 = i) \) for event \( A \)

\[
\mathbb{E}_i(Z) \overset{\text{def}}{=} \mathbb{E}(Z|X_0 = i) = \sum_z z \cdot \mathbb{P}(Z = z|X_0 = i) \text{ for real r.v. } Z (\text{some integral})
\]
Transition matrix & graph

\[
P = \text{stochastic matrix}:
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 0.5 & 0.25 & 0 & 0.25 & 0 \\
2 & 0.5 & 0 & 0.5 & 0 & 0 \\
3 & 0 & 0.8 & 0 & 0 & 0.2 \\
4 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 0.33 & 0 & 2/3 & 0 \\
\end{array}
\]

- positive coeff: \( \forall i, j, p_{ij} \geq 0 \)
- \( \sum \) over line = 1: \( \forall i, \sum_j p_{ij} = 1 \)

MC = "random walk": realization \( X_0(\omega), X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega), X_5(\omega), \ldots \) : walk in the transition graph

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Markov property in practice

Theorem (“General” Markov property)

Let \((X_n)\) MC with values in \(E\), at time \(n \in \mathbb{N}\) in state \(i \in E\),
let \(I^+ \in \mathcal{P}(E)^{\otimes \mathbb{N}}\) a set of trajectories in the future,
let \(I^- \in \mathcal{P}(E^n)\) a set of trajectories in the past,
\[\mathbb{P}((X_{n+1}, X_{n+2}, \ldots) \in I^+ | (X_0, \ldots, X_{n-1}) \in I^-, X_n = i) = \mathbb{P}((X_{n+1}, X_{n+2}, \ldots) \in I^+ | X_n = i)\]
And if homogeneous MC, this term is :
\[\mathbb{P}((X_1, X_2, \ldots) \in I^+ | X_0 = i)\]

English formulation : \(\forall i \in E, \forall n \in \mathbb{N}\), the future at time \(n\) and the past at time \(n\) are conditionally independent given the present state \(X_n = i\).

Examples of use :
\[\mathbb{P}(X_{10} = a, X_7 = b | X_5 = c, X_3 = d, X_2 = e) = \mathbb{P}(X_{10} = a, X_7 = b | X_5 = c)\]
\[\mathbb{P}(\forall n \geq 11, X_n \not\in \{a, b\} | X_{10} = c, \forall n \leq 9, X_n \in \{d, e\}) = \mathbb{P}(\forall n \geq 11, X_n \not\in \{a, b\} | X_{10} = c)\]
Chapman-Kolmogorov Equations (I)

**Notation:** \( p_{ij}(r, r+s) \stackrel{\text{def}}{=} \mathbb{P}(X_{r+s} = j \mid X_r = i) \) for \( i, j \in E, r, s \in \mathbb{N} \).

**Theorem (Chapman-Kolmogorov)**

Any MC \((X_n)_{n \in \mathbb{N}}\) satisfies the equations:

\[
p_{ij}(r, r+s+t) = \sum_k p_{ik}(r, r+s)p_{kj}(r+s, r+s+t)
\]

**Corollary (Matrix version)**

Given matrices \(P(r, r+s) \stackrel{\text{def}}{=} \left(p_{ij}(r, r+s)\right)_{i,j \in E},\) then \( \forall r, s, t \in \mathbb{N}, \)

\[
P(r, r+s+t) = P(r, r+s)P(r+s, r+s+t)
\]

**Corollary (Homogeneous case)**

If HMC, *prob to jump from \( i \) to \( j \) in \( n \) steps = coeff \( i,j \) of \( P^n \) denoted \( p_{ij}(n). \)
Chapman-Kolmogorov Equations (II)

Vector notation of the law $\nu$ of a r.v. $X$ with values in $E$:
$$\nu = (\nu_i)_{i \in E}$$
line vector with $\nu_i \overset{\text{def}}{=} \mathbb{P}(X = i)$

Corollary (Homogeneous case)

If HMC, the law $\pi^{(n)}$ of $X_n$ is fully set by the matrix $P$ and the law $\pi^{(0)}$ of $X_0$:
$$\pi^{(n)} = \pi^{(0)} P^n.$$
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**Vector notation of the law** \( \nu \) **of a r.v.** \( X \) **with values in** \( E \) :
\[
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\]

**Corollary (Homogeneous case)**

*If HMC, the law* \( \pi^{(n)} \) **of** \( X_n \) **is fully set by the matrix** \( P \) **and the law** \( \pi^{(0)} \) **of** \( X_0 : \pi^{(n)} = \pi^{(0)} P^n. \)
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line vector with $\nu_i \stackrel{\text{def}}{=} \mathbb{P}(X = i)$

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*If HMC, the law $\pi^{(n)}$ of $X_n$ is fully set by the matrix $P$ and the law $\pi^{(0)}$ of $X_0$ : $\pi^{(n)} = \pi^{(0)} P^n$.***
Chapman-Kolmogorov Equations (II)

Vector notation of the law $\nu$ of a r.v. $X$ with values in $E$:
$$\nu = (\nu_i)_{i \in E} \text{ line vector with } \nu_i \overset{\text{def}}{=} P(X = i)$$

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If HMC, the law $\pi^{(n)}$ of $X_n$ is fully set by the matrix $P$ and the law $\pi^{(0)}$ of $X_0$:
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If HMC, the law $\pi^{(n)}$ of $X_n$ is fully set by the matrix $P$ and the law $\pi^{(0)}$ of $X_0$:

$$\pi^{(n)} = \pi^{(0)} P^n.$$
Definition (Stopping time of a stochastic process)

Stopping time $T$ of stochastic process $(X_n)_{n \in \mathbb{N}}$: random variable with values in $\mathbb{N} \cup \{+\infty\}$ such that for all $n \in \mathbb{N}$, event $\{T = n\}$ can be described using $X_0, \ldots, X_n$: $\{T = n\} = \{(X_0, \ldots, X_n) \in I\}$ for a set of trajectories $I \subseteq E^{n+1}$.

Intuition: time event which can be expressed with no reference to the future.

Examples: let $(X_n)$ MC with values in $E$ and $F \subseteq E$,

- Time to reach $F$: $\tau_F = \inf\{n \geq 0 | X_n \in F\}$?
- Time to come back to $F$: $T_F = \inf\{n \geq 1 | X_n \in F\}$?
- Last time in $F$: $L_F = \sup\{n \geq 0 | X_n \in F\}$?
Stopping time: definition & examples

Definition (Stopping time of a stochastic process)

Stopping time $T$ of stochastic process $(X_n)_{n \in \mathbb{N}}$: r.v. with values in $\mathbb{N} \cup \{+\infty\}$ s.t. $\forall n \in \mathbb{N}$, event $\{T = n\}$ can be described using $X_0, \ldots, X_n$: $\{T = n\} = \{(X_0, \ldots, X_n) \in I\}$ for a set of trajectories $I \subseteq E^{n+1}$.

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Stopping time: definition & examples

Definition (Stopping time of a stochastic process)

A stopping time \( T \) of stochastic process \((X_n)_{n \in \mathbb{N}}\) is a random variable with values in \(\mathbb{N} \cup \{+\infty\}\) such that for every \(n \in \mathbb{N}\), the event \(\{T = n\}\) can be described using \(X_0, \ldots, X_n\): \(\{T = n\} = \{(X_0, \ldots, X_n) \in I\}\) for some set of trajectories \(I \subseteq E^{n+1}\).

Intuition: time event which can be expressed with no reference to the future.

Examples: let \((X_n)\) MC with values in \(E\) and \(F \subseteq E\),
- Time to reach \(F\): \(\tau_F = \inf\{n \geq 0 | X_n \in F\}\)
- Time to come back to \(F\): \(T_F = \inf\{n \geq 1 | X_n \in F\}\)
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Stopping time: definition & examples

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*Stopping time $T$ of stoch proc $(X_n)_{n \in \mathbb{N}}$: r.v. with values in $\mathbb{N} \cup \{+\infty\}$ s.t. $\forall n \in \mathbb{N}$, event $\{T = n\}$ can be described using $X_0, \ldots, X_n$: $\{T = n\} = \{(X_0, \ldots, X_n) \in I\}$ for a set of trajectories $I \subseteq E^{n+1}$.*

**Intuition**: time event which can be expressed with no reference to the future.

**Examples**: let $(X_n)$ MC with values in $E$ and $F \subseteq E$,

- Time to reach $F$: $\tau_F = \inf\{n \geq 0 | X_n \in F\}$ ☑
- Time to come back to $F$: $T_F = \inf\{n \geq 1 | X_n \in F\}$ ☑
- Last time in $F$: $L_F = \sup\{n \geq 0 | X_n \in F\}$ ☹️
Definition (Stopping time of a stochastic process)

**Stopping time** $T$ of stochastic process $(X_n)_{n \in \mathbb{N}}$ is a random variable with values in $\mathbb{N} \cup \{+\infty\}$ such that for all $n \in \mathbb{N}$, the event $\{T = n\}$ can be described using $X_0, \ldots, X_n$ as $\{T = n\} = \{(X_0, \ldots, X_n) \in I\}$ for a set of trajectories $I \subset E^{n+1}$.

**Intuition**: time event which can be expressed with no reference to the future.

**Examples**: let $(X_n)$ MC with values in $E$ and $F \subseteq E$,

- Time to reach $F$: $\tau_F = \inf\{n \geq 0 | X_n \in F\}$
- Time to come back to $F$: $T_F = \inf\{n \geq 1 | X_n \in F\}$
- Last time in $F$: $L_F = \sup\{n \geq 0 | X_n \in F\}$

**Special notation**: for $i \in E$, $T_i \overset{\text{def}}{=} T\{i\}$ and $\tau_i \overset{\text{def}}{=} \tau\{i\}$.
Exercise: let $T, T_1, T_2$ stopping times for $(X_n)$, tell whether the next r.v. are also stopping times for $(X_n)$?

1. a constant r.v. $c$
2. $T + c$ where $c \in \mathbb{N}^*$ fixed
3. $T - c$ where $c \in \mathbb{N}^*$ fixed
4. $\min(T_1, T_2)$
5. $\max(T_1, T_2)$
6. $N(t) = \max\{n \in \mathbb{N}|X_0 + X_1 + \ldots + X_n \leq t\}$ ($X_n$ positive r.v.)
7. $N(t) + 1$
**Theorem (Strong Markov property)**

- Let $T$ stopping time for HMC $(X_n)$, then subject to $T < +\infty$ and $X_T = i$, $(X_{T+n})_{n\geq 0}$ is markovian and independent of $X_0, \ldots, X_T$ (also denoted $(X_{T\wedge n})_{n\geq 0}$ où $\wedge = \min$).
- Moreover, for any event $A$ described with $X_0, \ldots, X_T$ and $I^+ \in \mathcal{P}(E)^\otimes \mathbb{N}$
  \[
  \mathbb{P}((X_{T+1}, X_{T+2}, \ldots) \in I^+ | X_T = i, T < +\infty, A) = \mathbb{P}((X_1, X_2, \ldots) \in I^+ | X_0 = i)
  \]

**Intuition** : starting to look at some HMC from a stopping time $T$ = reset counters to zero

⚠️ if $T$ not a stopping time, risk to lose this property (cf TD).
⚠️ if MC not homogeneous, risk to lose this property (even if $T$ stopping time).
“One step forward” method: small step without strong Markov (I)

**Example**: probability $P_i(\tau_F < +\infty)$ to reach a set $F$ of states starting from state $i$

**Application**: non biased walk over $\{0, \ldots, N\}$ where 0, $N$ absorbing
"One step forward" method: small step without strong Markov (I)

**Example**: probability \( P_i(\tau_F < +\infty) \) to reach a set \( F \) of states starting from state \( i \)

**Proposition**

The values \( h_i = P_i(\tau_F < +\infty) \) form the minimum positive solution in \( \mathbb{R} \) of the linear system:

\[
\begin{cases}
  h_i = 1 & \text{for all } i \in F \\
  h_i = \sum_{j \in E} p_{ij} h_j & \text{for all } i \notin F
\end{cases}
\]

**Application**: non biased walk over \( \{0, \ldots, N\} \) where 0, \( N \) absorbing

![Diagram of a non biased walk over \( \{0, \ldots, N\} \) with states 0, 1, ..., N, and transition probabilities.

\( 1 \)
\( 0 \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
\( \frac{1}{2} \)
“One step forward” method: small step without strong Markov (I)

Example: probability \( P_i(\tau_F < +\infty) \) to reach a set \( F \) of states starting from state \( i \)

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\end{align*}
\]

Application: non biased walk over \( \{0, \ldots, N\} \) where 0, \( N \) absorbing

\[ \rightarrow \text{probability to be absorbed by 0 starting from } i = (N - j)/N \]
“One step forward” method: small step with strong Markov (II)

**Example**: Mean time $\mathbb{E}_i(\tau_F)$ to reach a set $F$ of states starting from state $i$

**Application**: 1D non biased walk over $\{0, \ldots, N\}$ with $F = \{0, N\}$. 
"One step forward" method: small step with strong Markov

**Example**: Mean time $E_i(\tau_F)$ to reach a set $F$ of states starting from state $i$

**Proposition**

The values $t_i = E_i(\tau_F)$ form the min positive solution in $\mathbb{R} \cup \{\infty\}$ of the linear system:

$$
\begin{align*}
    t_i &= 0 & \text{for tout } i \in F \\
    t_i &= 1 + \sum_{j \notin F} p_{ij} t_j & \text{pour tout } i \notin F
\end{align*}
$$

**Application**: 1D non biased walk over $\{0, \ldots, N\}$ with $F = \{0, N\}$.
“One step forward” method: small step with strong Markov property (II)

Example: Mean time $\mathbb{E}_i(\tau_F)$ to reach a set $F$ of states starting from state $i$.

Proposition

The values $t_i = \mathbb{E}_i(\tau_F)$ form the min positive solution in $\mathbb{R} \cup \{\infty\}$ of the linear system:

$$
\begin{align*}
    t_i &= 0 & \text{pour tout } i \in F \\
    t_i &= 1 + \sum_{j \not\in F} p_{ij} t_j & \text{pour tout } i \not\in F
\end{align*}
$$

Application: 1D non biased walk over $\{0, \ldots, N\}$ with $F = \{0, N\}$.

$\rightarrow$ mean time to reach 0 or $N$ starting from $i = i(N - i)$.
Example: law of nb of visits to state $i$ given reaching probabilities, for HMC $(X_n)$.

Lemma (nb of visits to a state & probas of access between states)

Let $N_i \overset{\text{def}}{=} \sum_{n=1}^{+\infty} \mathbb{1}_{X_n=i}$ nb of visits to $i$ from time 1,
Let $f_{ij} \overset{\text{def}}{=} \mathbb{P}_i(T_j < \infty)$ proba de reach $j$ after leaving $i$,
Then:

$$
\mathbb{P}_j(N_i = n) = \begin{cases} 
    f_{ji} f_{ii}^{n-1} (1 - f_{ii}) & \text{if } n \geq 1 \\
    1 - f_{ji} & \text{if } n = 0
\end{cases}
$$

Corollary (returns to the same state)

If $f_{ii} = 1$, then $\mathbb{P}_i(N_i = \infty) = 1$ et $\mathbb{E}_i(N_i) = +\infty$.
If $f_{ii} < 1$, then $\mathbb{P}_i(N_i = \infty) = 0$ et $\mathbb{E}_i(N_i) = f_{ii} / (1 - f_{ii}) < +\infty$. 
Irreducibility: definitions

Definition (Communication in HMC)

Two states $i$ et $j$ communicate if there exist a path from $i$ to $j$ and a path from $j$ to $i$ in the transition graph.

Proposition (Classes of communication)

Communication = equivalence relation partitioning states into equivalence classes, called classes of communication (= strongly connected components of the transition graph).

Definition (Irreducible HMC)

HMC is irreducible if it has only one class of communication (i.e. strongly connected transition graph).
Proposition (Bags with no cycle)

Let $G$ directed graph, with strongly connected components $C_1, \ldots, C_p$, then its quotient graph (for strong connection relation) defined by $\langle G \rangle = G / C_1 / \ldots / C_p$ (contraction of each component into one vertex) is acyclic.

Definition (Closed/final/absorbing class)

Class of communication is closed/final/absorbing if all states reachable from this class remain in this class (“maximal” strongly connected comp. in the quotient graph).
Irreducibility: example

⚠ if nb $\infty$ of states, one may see $\infty$ classes or classes $\infty$. 
Irreducibility: example

⚠ if nb $\infty$ of states, one may see $\infty$ classes or classes $\infty$. 
Irreducibility: example

⚠️ if nb $\infty$ of states, one may see $\infty$ classes or classes $\infty$. 
Irreducibility: example

⚠️ if nb $\infty$ of states, one may see $\infty$ classes or classes $\infty$. 

composantes fortement connexes
If nb $\infty$ of states, one may see $\infty$ classes or classes $\infty$. 
Irreducibility: example

⚠️ if nb $\infty$ of states, one may see $\infty$ classes or classes $\infty$. 
Periodicity: definitions

Definition (Period of a state in HMC)

State $i$ has period $d_i \overset{\text{def}}{=} \gcd\{n \geq 1 | p_{ii}(n) > 0\}$ (i.e. GCD lengths of cycles traversing $i$ in the transition graph).

Proposition (Irreducibility & periodicity)

In a class of communication (strong. conn. comp.), all states have the same period.

Definition (Period of an irreducible HMC)

- Period of irred HMC: period common to all its states ($= \text{PGCD lengths of all cycles in transition graph}$.)
- Aperiodic irred HMC: if period $= 1$. 
**Exercise**: find the period of those graphs.
**Exercise**: find the period of those graphs.

### Periodicity: examples

- **A**
  - Graph: Cycle of 4 vertices ($\alpha$, $\beta$, $\gamma$, $\delta$)
  - Period: 1

- **B**
  - Graph: Cycle of 3 vertices ($a$, $b$, $c$)
  - Period: 1

- **C**
  - Graph: Chain of 8 vertices ($1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$)
  - Period: 1
Exercise: find the period of those graphs.

A
period = 1

B
period = 3

C
Periodicity: examples

Exercise: find the period of those graphs.

- **A**: period = 1
- **B**: period = 3
- **C**: period = 1
Periodicity: structure

Theorem (cycle of bags)

Let $G$ strongly connected directed graph of period $d$, then there exists a partition $V_0, \ldots, V_{d-1}$ of vertices such that any edge leaving $V_p$ reaches $V_{p+1}$ (with the convention $V_{d+1} = V_0$).
Theorem (cycle of bags)

Let $G$ strongly connected directed graph of period $d$, then there exists a partition $V_0,\ldots,V_{d-1}$ of vertices such that any edge leaving $V_p$ reaches $V_{p+1}$ (with the convention $V_{d+1} = V_0$).
Invariance : definitions

Framework : \((X_n)\) HMC with transition matrix \(P\).

Definition (Invariant/stationary measure)

Invariant/stationary measure for \(P\) : \(\mu = (\mu_i)_{i \in E} \in \mathbb{R}^E\) such that \(\mu \geq 0, \mu \neq 0\) and \(\mu P = \mu\), i.e. \(\forall i \mu_i \geq 0, \exists i \mu_i \neq 0\) and \(\sum_j \mu_j p_{ji} = \mu_i\).

Definition (Invariant/stationary probability distribution)

Inv./stat. distribution for \(P\) : invariant measure \(\mu\) with \(\sum_{i \in E} \mu_i < +\infty\). In this case, renormalized \(\pi = (\pi_i)_{i \in E}\) with \(\pi_i = \mu_i / \sum_{j \in E} \mu_j\) is called invariant/stationary probability distribution (\(\sum_{i \in E} \pi_i = 1\)).

Terminology : if law of \(X_n =\) invariant proba distrib, the process is said to be “in stationary regime”, “at equilibrium” ...
Invariance : structure

**Exercise** : how many invariant proba distrib for an HMC?

- 0
- 1
- \( \text{nb fini} \geq 2 \)
- \( \infty \)
**Exercise**: how many invariant proba distrib for an HMC?

- 0
- 1
- nb fini ≥ 2
- ∞
Invariance : structure

Exercise : how many invariant proba distrib for an HMC?

0
1
nb fini ≥ 2
∞
Exercise: how many invariant proba distrib for an HMC?

- 0
- 1
- nb fini ≥ 2
- ∞
Exercise: how many invariant proba distrib for an HMC?

0

1

nb fini ≥ 2

∞

Theorem (structure of invariant proba distrib)

The invariant proba distrib of an HMC form a convex polyhedron in \( \mathbb{R}^E_+ \): it is the convex hull of the invariant proba distrib of final classes of communication.
Recurrence: definitions

Definition (transitory/recurrent null/positive state)

Let \((X_n)\) HMC with values in \(E\) and \(T_i\) time to return to \(i\),
- state \(i\) transitory if \(P_i(T_i < +\infty) < 1\),
- state \(i\) recurrent if \(P_i(T_i < +\infty) = 1\),
- state \(i\) null recurrent if \(P_i(T_i < +\infty) = 1\) but \(E_i(T_i) = +\infty\),
- state \(i\) positive recurrent if \(E_i(T_i) < +\infty\) thus \(P_i(T_i < +\infty) = 1\).

Proposition (finite return time ⇔ infinite nb of visits)

state \(i\) recurrent ⇔ \(P_i(\infty \text{ nb of visits of } i)=1\) ⇔ \(E_i(\text{nb of visits of } i)=+\infty\)
state \(i\) transitory ⇔ \(P_i(\text{finite nb of visits of } i)=1\) ⇔ \(E_i(\text{nb of visits of } i)<+\infty\)

Corollary (potential matrix criterium)

\[ i \text{ recurrent iff } \sum_{n=0}^{+\infty} p_{ii}(n) = +\infty \]
Proposition

In a class of communication (strong. conn. comp.) of an HMC, the states are either all recurrent, or all transitory.
If the are recurrent, the class is closed and $\forall j, \mathbb{P}(T_j < +\infty) = 1$.

Corollary

An irreducible chain is either recurrent (all states are recurrent), or transitory (all states are transitory).

Question: HMC irreducible $\Rightarrow$ HMC recurrent?
Irreducibility & Recurrence

**Proposition**

In a class of communication (strong. conn. comp.) of an HMC, the states are either all recurrent, or all transitory. If they are recurrent, the class is closed and $\forall j$, $\mathbb{P}(T_j < +\infty) = 1$.

**Corollary**

An irreducible chain is either recurrent (all states are recurrent), or transitory (all states are transitory).

**Question** : HMC irreducible $\Rightarrow$ HMC recurrent? NO!

**Contrex** : 1D walk space homogeneous, recurrent iff $p = 1/2$

(compute $p_{00}(n)$ explicitly then estimate $\sum_{n=0}^{+\infty} p_{ii}(n)$ with Stirling)
Invariance & Recurrence

Theorem (if irreducible, recurrence $\Rightarrow$ invariant measure)

Let $(X_n)$ HMC irred and recurrent, of transition matrix $P$, Let state $0$ fixed arbitrarily and $T_0$ time to return to $0$, Let $V_i \overset{\text{def}}{=} \sum_{n=1}^{T_0} \mathbb{1}_{X_n=i}$ nb of visits of $i$ between time $0$ (excluded) and return time $T_0$ (included), define $x_i \overset{\text{def}}{=} E_0[V_i]$ average nb of visits of $i$ between two visits of $0$. Then:

1. $0 < x_i < \infty$ for all $i \in E$
2. $(x_i)_{i \in E}$ invariant measure of $P$ (canonical inv measure for $0$)
3. $P$ admits an unique invariant measure up to a constant factor

⚠️ HMC irreducible, with invariant measure $\Rightarrow$ HMC recurrent?
Invariance & Recurrence

Theorem (if irreducible, recurrence $\Rightarrow$ invariant measure)

Let $(X_n)$ HMC irred and recurrent, of transition matrice $P$,
Let state 0 fixed arbitrarily and $T_0$ time to return to 0,
Let $V_i \overset{\text{def}}{=} \sum_{n=1}^{T_0} 1_{X_n=i}$ nb of visits of $i$ between time 0 (excluded) and return time $T_0$ (included), define $x_i \overset{\text{def}}{=} \mathbb{E}_0[V_i]$ average nb of visits of $i$ between two visits of 0. Then:

1. $0 < x_i < \infty$ for all $i \in E$
2. $(x_i)_{i \in E}$ invariant measure of $P$ (canonical inv measure for 0)
3. $P$ admits an unique invariant measure up to a constant factor

⚠ HMC irreducible, with invariant measure $\Rightarrow$ HMC recurrent?

NO! look again 1D space homogeneous random walk, $p \neq 1/2$, they admit $1 = (\ldots, 1, 1, 1, \ldots)$ as invariant measure
Invariance & Positive recurrence

Theorem (if irreducible, positive recurrence $\iff$ inv proba distrib)

Let $(X_n)$ HMC irreducible, of transition matrix $P$, we have the equivalence:

1. $(X_n)$ admits a positive recurrent state,
2. $(X_n)$ has all its states positive recurrent,
3. $(X_n)$ admits an invariant proba distribution.

In this case, the invariant proba distrib $\pi = (\pi_i)$ is unique and satisfies $\pi_i = 1/\mathbb{E}_i(T_i) > 0$ where $T_i$ time to return to $i$. The chain is called positive recurrent.

Ex of HMC irreducible recurrent but not positive recurrent?
Theorem (if irreducible, positive recurrence $\Leftrightarrow$ inv proba distrib)

Let $(X_n)$ HMC irred, of transition matrix $P$, we have the equivalence:

1. $(X_n)$ admits a positive recurrent state,
2. $(X_n)$ has all its states positive recurrent,
3. $(X_n)$ admits an invariant proba distribution.

In this case, the invariant proba distrib $\pi = (\pi_i)$ is unique and satisfies $\pi_i = 1/\mathbb{E}_i(T_i) > 0$ where $T_i$ time to return to $i$. The chain is called positive recurrent.

Ex of HMC irred recurrent but not positive recurrent? YES, e.g. symmetric random walk over $\mathbb{Z}$!
Special case: HMC with finite nb of states

**Proposition**

*any finite state irreducible HMC is positive recurrent.*

**Theorem (Perron 1907 - Frobenius 1912)**

Let $P$ transition matrix of irred HMC, with $N$ states, with period $d$, with sorted complex eigenvalues $|\lambda_1| \geq \ldots \geq |\lambda_N|$ then

1. $\lambda_1 = 1$ eigenvalue of $P$,
2. complex unit roots $\lambda_1 = \omega^0, \lambda_2 = \omega^1, \ldots, \lambda_d = \omega^{d-1}$ où $\omega = e^{2\pi i/d}$, are eigenvalues of $P$,
3. other eigenvalues $\lambda_{d+1}, \ldots, \lambda_N$ satisfy $|\lambda_j| < 1$.

**Corollary (irred and aperiodic HMC)**

$$P^n = 1^T \pi + O(n^{m_2-1} |\lambda_2|^n) \text{ where } m_2 \text{ multiplicity of } \lambda_2 \text{ (}|\lambda_2| < 1|)$$
Asymptotic convergence:

**Theorem (Convergence in law for HMC)**

Let \((X_n)\) HMC irreducible, positive recurrent, aperiodic, of transition matrix \(P\) and stationary distribution \(\pi\). Then for any initial distribution \(\nu\), for any state \(i\),

\[
\lim_{n \to +\infty} P(X_n = i) = \pi_i
\]

More precisely, \(\lim_{n \to +\infty} ||\nu P^n - \pi||_\infty = 0\).

A classical proof: by coupling Markov chains

\(\triangle\) Essential hypothesis: period = 1.
Theorem (Ergodicity for HMC)

Let \((X_n)\) HMC with values in \(E\), irred, positive recurrent of invariant distrib \(\pi\), and let \(f : E \to \mathbb{R}\) such that \(\sum_{i \in E} |f(i)|\pi_i < \infty\), then for any initial law \(\nu\), almost surely,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \sum_{i \in E} f(i)\pi_i
\]
Question: dealing with irred HMC of period \( d \geq 2 \)?
Periodic irreducible case

**Question**: dealing with irred HMC of period \( d \geq 2 \)?

**Reductions**: return to aperiodic case with \( \frac{1+P+\ldots+P^{d-1}}{d} \) or \( P^d \)

**Theorem (Convergence - periodic case)**

Let \((X_n)\) HMC irreducible, positive recurrent, of period \( d \), with transition matrix \( P \), let \( V_0, \ldots, V_{d-1} \) the bag cycle partition. Then for any initial distribution \( \nu \), for all \( 0 \leq r \leq d-1 \), for any state \( i \in V_r \),

\[
\lim_{n \to +\infty} P(X_{nd+r} = i) = \frac{d}{\mathbb{E}_i(T_i)}
\]

More precisely, \( \lim_{n \to +\infty} \|\nu P^{nd+r} - \frac{d}{\mathbb{E}_i(T_i)}\|_\infty = 0 \)
Periodic irreducible case

Question: dealing with irred HMC of period \( d \geq 2 \)?

Reductions: return to aperiodic case with \( \frac{I+P+\ldots+P^{d-1}}{d} \) or \( P^d \) \( \triangleleft \) loosing irred

Theorem (Convergence - periodic case)

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\[
\lim_{n \to +\infty} \mathbb{P}(X_{nd+r} = i) = \frac{d}{\mathbb{E}_i(T_i)}
\]

More precisely, \( \lim_{n \to +\infty} \|\nu P^{nd+r} - \frac{d}{\mathbb{E}_i(T_i)}\|_\infty = 0 \)
Non irreducible case

Asymptotic study of the general case

- Study the transition graph structure and identify final classes
- Study the absorption probabilities of each final class
- Study the asymptotic behaviour in each final class (period, recurrence, invariant distribution ...
How to decide irreducibility? aperiodicity?

Transition graph structure:

- Computing strongly connected comp and acyclic quotient graph: computable in general (depends on the chain description if nb states $\infty$), linear in time and space (if finite nb states) $\rightarrow$ algs based on DFS (Tarjan 1972, Kosaraju 1978)

- Computing the period: computable in general (depends on the chain description if nb states $\infty$), linear in time and space (if finite nb states) $\rightarrow$ algo based on graph searching (Denardo 1977)
Definition (Face-homogeneous HMC over $\mathbb{N}^d$ with unit jumps)

For all $\Lambda \subseteq \{1, \ldots, d\}$, HMC ($X_n$) space homogeneous over the face $\mathbb{N}_\Lambda \overset{\text{def}}{=} \{ x = (x_1, \ldots, x_d) \in \mathbb{N}^d \mid \forall \lambda \in \Lambda, x_\lambda = 0, \forall \lambda \notin \Lambda, x_\lambda > 0 \}$ such that $\forall \Delta \in \{-1, 0, +1\}^d$, $\forall x \in \mathbb{N}_\Lambda$, proba to jump from $x$ to $x + \Delta$ is $p(\Lambda, \Delta)$, with $\sum_{\Delta \in \{-1, 0, +1\}^d} p(\Lambda, \Delta) = 1$.
**How to decide recurrence?**

**Definition (Face-homogeneous HMC over \( \mathbb{N}^d \) with unit jumps)**

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face \( \mathbb{N}_\Lambda \) for \( \Lambda = \{1, 2\} \)
How to decide recurrence?

**Definition (Face-homogeneous HMC over \( \mathbb{N}^d \) with unit jumps)**

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\[
\mathbb{N}_\Lambda \overset{\text{def}}{=} \{ x = (x_1, \ldots, x_d) \in \mathbb{N}^d | \forall \lambda \in \Lambda, x_\lambda = 0, \ \forall \lambda \not\in \Lambda, x_\lambda > 0 \} \text{ such that } \forall \Delta \in \{-1, 0, +1\}^d, \ \forall x \in \mathbb{N}_\Lambda, \text{ proba to jump from } x \text{ to } x + \Delta \text{ is } p(\Lambda, \Delta), \text{ with } \sum_{\Delta \in \{-1, 0, +1\}^d} p(\Lambda, \Delta) = 1.
\]

face \( \mathbb{N}_\Lambda \) for \( \Lambda = \{1, 3\} \)
How to decide recurrence?

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face $\mathbb{N}_\Lambda$ for $\Lambda = \{2,3\}$
How to decide recurrence?

**Definition (Face-homogeneous HMC over $\mathbb{N}^d$ with unit jumps)**

For all $\Lambda \subseteq \{1, \ldots, d\}$, HMC $(X_n)$ space homogeneous over the face $\mathbb{N}_\Lambda \overset{\text{def}}{=} \{x = (x_1, \ldots, x_d) \in \mathbb{N}^d | \forall \lambda \in \Lambda, x_\lambda = 0, \forall \lambda \notin \Lambda, x_\lambda > 0\}$ such that $\forall \Delta \in \{-1, 0, +1\}^d$, $\forall x \in \mathbb{N}_\Lambda$, proba to jump from $x$ to $x + \Delta$ is $p(\Lambda, \Delta)$, with $\sum_{\Delta \in \{-1, 0, +1\}^d} p(\Lambda, \Delta) = 1$.

face $\mathbb{N}_\Lambda$ for $\Lambda = \{1\}$
How to decide recurrence?

Definition (Face-homogeneous HMC over $\mathbb{N}^d$ with unit jumps)

For all $\Lambda \subseteq \{1, \ldots, d\}$, HMC $(X_n)$ space homogeneous over the face $\mathbb{N}_\Lambda \overset{\text{def}}{=} \{x = (x_1, \ldots, x_d) \in \mathbb{N}^d | \forall \lambda \in \Lambda, x_\lambda = 0, \forall \lambda \notin \Lambda, x_\lambda > 0\}$ such that $\forall \Delta \in \{-1, 0, +1\}^d, \forall x \in \mathbb{N}_\Lambda$, proba to jump from $x$ to $x + \Delta$ is $p(\Lambda, \Delta)$, with $\sum_{\Delta \in \{-1, 0, +1\}^d} p(\Lambda, \Delta) = 1$. 

face $\mathbb{N}_\Lambda$ for $\Lambda = \{2\}$
How to decide recurrence?

Definition (Face-homogeneous HMC over $\mathbb{N}^d$ with unit jumps)

For all $\Lambda \subseteq \{1, \ldots, d\}$, HMC $(X_n)$ space homogeneous over the face $\mathbb{N}_\Lambda \overset{\text{def}}{=} \{ x = (x_1, \ldots, x_d) \in \mathbb{N}^d | \forall \lambda \in \Lambda, x_\lambda = 0, \ \forall \lambda \notin \Lambda, x_\lambda > 0 \}$ such that $\forall \Delta \in \{-1, 0, +1\}^d$, $\forall x \in \mathbb{N}_\Lambda$, proba to jump from $x$ to $x + \Delta$ is $p(\Lambda, \Delta)$, with $\sum_{\Delta \in \{-1, 0, +1\}^d} p(\Lambda, \Delta) = 1$.

face $\mathbb{N}_\Lambda$ for $\Lambda = \{3\}$
How to decide recurrence?

Definition (Face-homogeneous HMC over $\mathbb{N}^d$ with unit jumps)

For all $\Lambda \subseteq \{1, \ldots, d\}$, HMC $(X_n)$ space homogeneous over the face $\mathbb{N}_\Lambda \overset{\text{def}}{=} \{x = (x_1, \ldots, x_d) \in \mathbb{N}^d | \forall \lambda \in \Lambda, x_\lambda = 0, \forall \lambda \notin \Lambda, x_\lambda > 0\}$ such that $\forall \Delta \in \{-1,0,+1\}^d$, $\forall x \in \mathbb{N}_\Lambda$, proba to jump from $x$ to $x + \Delta$ is $p(\Lambda, \Delta)$, with $\sum_{\Delta \in \{-1,0,+1\}^d} p(\Lambda, \Delta) = 1$.

face $\mathbb{N}_\Lambda$ for $\Lambda = \emptyset$
How to decide recurrence?

Definition (Face-homogeneous HMC over $\mathbb{N}^d$ with unit jumps)

For all $\Lambda \subseteq \{1, \ldots, d\}$, HMC $(X_n)$ space homogeneous over the face

$$\mathbb{N}_\Lambda \overset{\text{def}}{=} \{x = (x_1, \ldots, x_d) \in \mathbb{N}^d | \forall \lambda \in \Lambda, x_\lambda = 0, \forall \lambda \not\in \Lambda, x_\lambda > 0\}$$

such that $\forall \Delta \in \{-1, 0, +1\}^d$, $\forall x \in \mathbb{N}_\Lambda$, proba to jump from $x$ to $x + \Delta$ is $p(\Lambda, \Delta)$, with $\sum_{\Delta \in \{-1, 0, +1\}^d} p(\Lambda, \Delta) = 1$.

HMC
face homogeneous
with unit jumps

Ex here: if $\Lambda = \{1, 2\}$ et
$\Delta = (+1, +1)$
$p(\Lambda, \Delta) = 1/2$
How to decide recurrence?

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For all \(\Lambda \subseteq \{1, \ldots, d\}\), HMC \((X_n)\) space homogeneous over the face \(\mathbb{N}_\Lambda \overset{\text{def}}{=} \{x = (x_1, \ldots, x_d) \in \mathbb{N}^d | \forall \lambda \in \Lambda, x_\lambda = 0, \forall \lambda \notin \Lambda, x_\lambda > 0\}\) such that \(\forall \Delta \in \{-1,0,+1\}^d, \forall x \in \mathbb{N}_\Lambda\), proba to jump from \(x\) to \(x + \Delta\) is \(p(\Lambda, \Delta)\), with \(\sum_{\Delta \in \{-1,0,+1\}^d} p(\Lambda, \Delta) = 1\).

**Theorem (Gamarnik 2002)**

Deciding for any \(d\) if HMC face-homogeneous over \(\mathbb{N}^d\) with unit jumps is positive recurrent, is undecidable.

**Theorem (Malyshev 1972, Menshikov 1974, Ignatyuk 1993)**

Deciding for fixed \(d \in \{1,2,3,4\}\) if HMC face-homogeneous over \(\mathbb{N}^d\) with unit jumps is positive recurrent, is decidable (open for fixed \(d \geq 5\)).
How to decide recurrence?

**Useful first step**: check irreducibility.

**Checking recurrence**:
- by returning to the definition (e.g. explicit value of $P_i(T_i < \infty)$)
- by the potential matrix criterium (nature of $\sum_{n\geq0} p_{ii}(n)$)

**Checking positive recurrence**:
- if finite nb states, obvious : yes iff irreducible
- by returning to the definition (e.g. explicit computation of $E_i(T_i)$)
- by searching a invariant distribution (search an inv measure & check at the end that $\sum_i \pi_i < \infty$),
- by the use of super/sub-martingales.
Martingales: definitions

Cond expectation of $Y$ real r.v. with respect to r.v. $X_n, ..., X_0$:

$\mathbb{E}(Y|X_n, ..., X_0) \overset{def}{=} \sum_{i_0, ..., i_n \in E} \mathbb{E}(Y|X_n = i_n, ..., X_0 = i_0) \mathbb{1}_{X_n = i_n, ..., X_0 = i_0} \triangle r.v.$

Definition (Martingale with respect to process $(X_n)_{n \in \mathbb{N}}$)

Process $(M_n)_{n \in \mathbb{N}}$ with real values martingale with respect to Process $(X_n)_{n \in \mathbb{N}}$ with values in $E$ if: $\forall n \in \mathbb{N}$, $\mathbb{E}|M_n| < \infty$ and $\mathbb{E}(M_{n+1}|X_n, ..., X_0) = M_n$. In this case, $\forall n \in \mathbb{N}$, $\mathbb{E}(M_n) = \mathbb{E}(M_0)$.

In practice: usually $M_n \overset{def}{=} f(X_n, ..., X_0)$, or even $f(X_n)$, then check if $\forall i_0, ..., i_n \in E$, $\mathbb{E}(M_{n+1}|X_n = i_n, ..., X_0 = i_0) = f(i_n, ..., i_0)$.

Example: $(X_n)$ symmetric walk over $\mathbb{Z}$, $M_n = f(X_n)$ with $f(i) = i$

Variants: sub-/super-martingale if $\forall n \in \mathbb{N}$,

$\mathbb{E}(M_{n+1}|X_n, ..., X_0) \geq M_n$ (resp $\leq$) and $\mathbb{E}|M_n| < \infty$
Martingales : stopping time theorem

**Theorem (Doob’s stopping theorem/ optional stopping theorem)**

Let \((M_n)\) martingale (resp. sub-/super-) for \((X_n)\) and \(T\) stopping time for \((X_n)\). If at least one of the next conditions is true:

1. \(T \leq N\) a.s. where \(N \in \mathbb{N}\)
2. \(T < \infty\) and \(\forall n \in \mathbb{N}, |M_n| \leq C\) a.s. where \(C \in \mathbb{R}_+\)
3. \(\mathbb{E}(T) < \infty\) and \(\forall n \in \mathbb{N}, |M_{n+1} - M_n| \leq C\) a.s. where \(C \in \mathbb{R}_+\)

Then \(\mathbb{E}(M_T) = \mathbb{E}(M_0)\) (resp. \(\geq/\leq\)).

**Applications** : \((X_n)\) symmetric walk over \(\mathbb{Z}\), \(0 \leq i \leq N\), let \(T = \tau_{\{0,N\}}\) absorption time by 0 or \(N\)

- Proba of absorption by \(N\) :
- Mean absorption time :
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- Proba of absorption by \(N\): \(M_n = X_n \Rightarrow \mathbb{P}_i(X_T = N) = i/N\)
- Mean absorption time:
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Applications: \((X_n)\) symmetric walk over \(\mathbb{Z}\), \(0 \leq i \leq N\), let \(T = \tau_{\{0,N\}}\) absorption time by 0 or \(N\)

1. Proba of absorption by \(N\): \(M_n = X_n \Rightarrow \mathbb{P}_i(X_T = N) = i/N\)
2. Mean absorption time: \(M_n = X_n^2 - 1 \Rightarrow \mathbb{E}_i(T) = i(N - i)\)
Theorem (one CS of positive recurrence - Foster 1953)

Let \((X_n)\) HMC irred with values in \(E\), if there exists \(h : E \rightarrow \mathbb{R}_+\), \(F\) fini \(\subseteq E\), \(\varepsilon > 0\) such that:

- \(\forall i \in F, \mathbb{E}_i(h(X_1)) = \sum_{j \in E} p_{ij} h(j) < \infty\), and
- \(\forall i \notin F, \mathbb{E}_i(h(X_1) - h(X_0)) = \sum_{j \in E} p_{ij} h(j) - h(i) \leq -\varepsilon\)

Then the chain is positive recurrent and \(\forall i \in F, \mathbb{E}_i(T_F) \leq h(i)/\varepsilon\).

Example: biased walk over \(\mathbb{N}\) with \(p < 1/2\)
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Example: biased walk over $\mathbb{N}$ with $p < 1/2$

\[
\begin{array}{ccc}
0 & \xrightarrow{1} & 1 \\
\downarrow{1-p} & & \downarrow{1-p} \\
& 2 & \xrightarrow{p} \\
& \downarrow{1-p} & \downarrow{1-p} \\
& 3 & \\
& \downarrow{1-p} & \xrightarrow{p} \\
\end{array}
\]

→ positive recurrent: take $F = \{0\}$ and $h(i) = i$
Martingales : Foster’s theorem (II)

Theorem (one CS of non positive recurrence - Tweedie 1976)

Let \((X_n)\) HMC irred with values in \(E\), if there exists \(h : E \to \mathbb{R}_+\), \(F\) finite \(\subseteq E\), \(c > 0\) such that:

- \(\forall i \in E, \mathbb{E}_i |h(X_1) - h(X_0)| = \sum_{j \in E} p_{ij} |h(j) - h(i)| \leq c\)
- \(\forall i \notin F, \mathbb{E}_i (h(X_1) - h(X_0)) = \sum_{j \in E} p_{ij} h(j) - h(i) \geq 0\)
- \(\exists i_0 \notin F, h(i_0) > \max_{i \in F} h(i)\)

Then the chain is not positive recurrent and \(\mathbb{E}_{i_0}(T_F) = +\infty\).

Example : biased walk over \(\mathbb{N}\) with \(p \geq 1/2\)

\[
\begin{array}{c}
0 \quad \xrightarrow{1-p} 1 \xrightarrow{p} 2 \xrightarrow{p} 3 \xrightarrow{1-p} \ldots \\
1-p \quad 1-p \quad 1-p \quad 1-p
\end{array}
\]
Martingales: Foster’s theorem (II)

Theorem (one CS of non positive recurrence - Tweedie 1976)

Let $(X_n)$ HMC irreducible with values in $E$, if there exists $h: E \to \mathbb{R}^+$, $F$ finite $\subseteq E$, $c > 0$ such that:

- $\forall i \in E$, $\mathbb{E}_i |h(X_1) - h(X_0)| = \sum_{j \in E} p_{ij} |h(j) - h(i)| \leq c$
- $\forall i \notin F$, $\mathbb{E}_i (h(X_1) - h(X_0)) = \sum_{j \in E} p_{ij} h(j) - h(i) \geq 0$
- $\exists i_0 \notin F$, $h(i_0) > \max_{i \in F} h(i)$

Then the chain is not positive recurrent and $\mathbb{E}_{i_0}(T_F) = +\infty$.

Example: biased walk over $\mathbb{N}$ with $p \geq 1/2$

\[
\begin{array}{cccc}
0 & \longrightarrow^1 & 1 & \longrightarrow^p 2 & \longrightarrow^p 3 & \longrightarrow^{1-p} \cdots \\ \\
\downarrow_{1-p} & & \downarrow_{1-p} & & \downarrow_{1-p} & \end{array}
\]

\[\rightarrow\text{ not positive recurrent : take } F = \{0\} \text{ and } h(i) = i \text{ or } h(i) = \mathbb{1}_{\geq 1}(i)\]
Invariant distribution: computation techniques

- Solve directly the linear system $\pi P = \pi$ with unknown $(\pi_i)_{i \in E}$ (combine/substitute, Gauss’ pivot, Cramer’s formulas ...).
- Introduce new linear equations using flow reasoning, to simplify the system solving.
- Pull out of the hat a good candidate, inject it in the linear system to check if it works, adjust its parameters if necessary.
Invariant distribution : flows

Proposition ("flow" vision of invariance)

Associate with distrib \( \pi = (\pi_i)_{i \in E} \) the flow \( f_{ij} \triangleq \pi_i p_{ij} \) from \( i \) to \( j \) for each edge \( ij \) in the transition graph. Then \( \pi \) inv distrib iff \( f \) satisfies Kirchoff's 1st law (preservation of the total flow at each state).

Proposition (Flow relations in the stationary regime)

Let \( \pi \) invariant distrib and \( S \subseteq E \), then:

\[
\sum_{i \not\in S} \pi_i p_{ij} = \sum_{j \in S} \pi_j p_{ji}
\]

Example : reversible Markov chains
Modeling steps with discrete time HMC

1. Define the space of states, list the states if possible
2. For each state, list events that may occur
3. Check whether the dynamics is Markovian, homogeneous for time and/or space

Examples: some models based on discrete time M/M/1 queues