Transformations and preservation of self-assembly dynamics through homotheties

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Abstract

We introduce a new notion in self-assembly, that of transforming the dynamics of assembly. This notion allows us to have transformation of the plane computed within the assembly process. Then we apply this notion to zooming, which gives us an impossibility result allowing us to pinpoint the notion of locality in the assembly process, and then a construction which allows us to make assembly somewhat scale-independent.

1 Introduction and definitions

Self-assembly, a concept introduced by Winfree in [6] and studied by Winfree, Rothemund and Adelmann ([5],[1],[4]) is a model of universal computation by DNA, and of other physical mechanism of accretion and crystallization. This model, consists of a soup with floating Wang tiles with glue on their sides which can stick to each other whenever there is enough glue.

It is a kind of dynamic version of Wang tiling with interesting properties as well as a quite realistic model of some physical or biological construction processes, which has been used experimentally in [4]. In particular, the scale at which the assembled patterned are observable is an important parameter. It can for example be linked to time-complexity[5].

In this article, we are interested in transforming self-assembled tilings by changing the set of tiles. We want these transformations to preserve the dynamics of the assembly, that is the different ways in which a shape can be assembled. This allows us to say that the transformation has been computed *by the tile-set itself* rather than computing the image of a shape by the transformation and then finding a tile-set that assembles into that image. More precisely, we are interested in homotheties (or zooming), which allows to change the scale of assembly.

These homotheties have been used in the literature ([8],[7],...) as tools to enhance the properties of the assemblies, making it robust to error and even self-healing. Yet, their interaction with complex assembly processes were not rigorously defined, as all the tilesets were deterministic. Our notion of homotheties has two uses. First, it is an algorithmic notion, allowing us to scale some tilesets, and a framework for applying a geometrical transformation to a tileset. But it is also natural enough to be used as a tool for comparison of tilesets: if the “macrotiles” are observed rather than designed, they characterize the relationship between two tilesets operating in the same manner, but at a different scale.

We first explain these notions on an example, then we define formally what is a *dynamics* and what it means to *scale* it, which gives a formal framework for the study of the assembly process itself and its properties such as speed, robustness and so on. This formal definition leads to an impossibility result: the dynamics of some self-assembling tiling can depend on its precise size, with locality effects such as information coming from opposite sides of a tile. Though, we are able to show that given Rothemund's RC-condition [4], or even the slightly more generic order condition, tilesets can indeed be scaled, as these effect are then impossible.

Our constructions, which can be extended to other transformations than homotheties, allow to define a notion of computation which is much more representative of what happens during the assembly.

1.1 The model of self-assembly

The model we study is based on Wang tiles, with some glue added to their sides. The model of self assembly is the following : at a given time, a tile can be added to a finite pattern if and only if its colors match these of its neighbors (like Wang tiles), and if the total strength of the bounds linking the tile to the pattern is more than a given parameter, called the *temperature*.

A Wang tile t on an alphabet Σ is an element of Σ^4. We will note them as follows: $t = (c_N(t), c_S(t), c_E(t), c_W(t))$.

We will use the usual direction functions on $\mathbb{Z}^2$: $N(x,y) = (x, y+1), S(x,y) = (x, y-1), E(x,y) = (x+1, y), W(x,y) = (x-1, y)$, and will note $-S = N, -N = S, -E = W, -W = E$. Given a pattern $p : \mathbb{Z}^2 \to T$, we will say that $N(x,y)$ is the northern neighbor of (x,y), and that $c_N(p(x,y))$ and $c_S(p(N(x,y)))$ are adjacent colors (or sides), and similarly for other directions. We use the usual notations for intervals to represent intervals of integers: $[a,b] = \{a, \dots, b\}$, $[a,b) = \{a, \dots, b-1\}$, and so on.

Let J be a set of Wang tiles. A *finite pattern* is a partial mapping from $\mathbb{Z}^2$ to J whose domain is finite and 4-connected. We call the domain of a pattern its *shape*.

If a pattern is compatible with the colors on the edges of the tiles (that is two adjacent edges always have the same color), we will say that it is a (finite) *configuration* of J.

Definition 1. *A self-assembly system is a 5-uplet $s = (\Sigma, g, T, t, seed)$ where*

- Σ *is a finite alphabet, the set of* colors*;*
- $g : (\Sigma \to \mathbb{N})$ *is called the* strength function*; for $c \in \Sigma$, we will say that $g(c)$ is the strength of the* glue $(c, g(c))$
- T *is a set of Wang tiles on the alphabet* Σ*, the* tile set*;*
- $t \in \mathbb{N}$ *is the* temperature*;*
- $seed \in T$ *is the* seed.

For graphical representations, we use the following conventions (see figure 1): each tile is represented by a square with symbols representing the glues on their side. The number of symbols on a side corresponds to the strength of the glue. The seed (marked by a star) will be the first finite pattern, which will inititate the growth process. The steps of this growth pattern are called transitions. A transition is the addition of a tile whose colors match those of a free slot in the pattern and whose bond are stronger than the temperature.

Definition 2 (transition)**.** *Given a system $s = (\Sigma, G, T, t, seed)$ and two configurations c and c', there is a* transition *between c and c', which we will note $c \mapsto c'$ if*

$$\exists(x*,y*),\ \begin{cases} shape(c') = shape(c) \cup \{(x*,y*)\} \\ \forall (x,y) \in shape(c), c(x,y) = c'(x,y) \\ \sum_{\{d\in\{N,S,E,W\} \mid d(x*,y*)\in shape(c)\}} g(c_d(c'(x*,y*))) \geq t \end{cases}$$

The configurations c such that there is a sequence of transitions from the initial pattern ($\{(0,0) \mapsto seed\}$*) to them are called* productions.

Note that we will limit ourselves to self-assembly systems with a temperature of 2, as is the case in the literature[1]. Most of the time, when there is no possible confusion, we use the terms tileset and system in an indifferent way, and leave the alphabet and the temperature (2) unspecified when their values are obvious from the context. We call direction of a transition the set $\{-d | d(x*,y*) \in c$ before the transition occurs.$\}$

Definition 3. *The dynamics of a self-assembled system is defined as the poset of the production, where the order relation is the transitive closure of transitions.*

Thus we have $c > c'$ (for c and c' two configurations) if and only if c is obtained from c' by adding one or several tiles according to the rules above.

Let us note that for any self-assembled system, its dynamics is an semi-lattice (that is, two elements always have a lower bound), and that for any production c, the set $\{x \leq c\}$ is a lattice. The proof of these two facts, stated in [3] is given in appendix.

This dynamics captures the way the assembly takes place, and allows us to reason on the parallelism and synchronization phenomena which take place during the assembly. The properties of the growth process are reflected by those of the lattice. For example, the speed of the growth in a given configuration is the outgoing degree of the vertex corresponding to that configuration. Confluent branches represent parallelism, and so on. For a detailled reference on (semi) lattices, see [2].

A *final production* is a maximal element of the dynamics, that is a production to which no tile can be added. Given a tileset T, we say that T *assembles* the set L_T of all its final productions.

2 Scaling while preserving dynamics

2.1 An example

When observing a tileset at larger scale, it is natural to consider each $s \times s$ square as a single tile, and to look at their interactions. We will call these squares *macrotiles*, and compare their interactions with those of the small scale tileset. Before any formal definition, we give an example of scaling, which will help to get the intuition of the kind of phenomenon we are going to capture.

In this section, we consider three tilesets. The first tileset, T, assembles a set of shapes $\mathcal{L}$. This set is the set of squares. The two other tilesets, F and D (for *same* F*inal productions* and *same* D*ynamics*) both assemble another set of shapes $\mathcal{L}'$, which is a scaled version of $\mathcal{L}$. We contrast F and D as D has an assembly process which is related to that of T, and not F.

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[1] temperature 1 self-assembly is rather trivial, and higher temperatures do not seem to be very different from temperature 2.

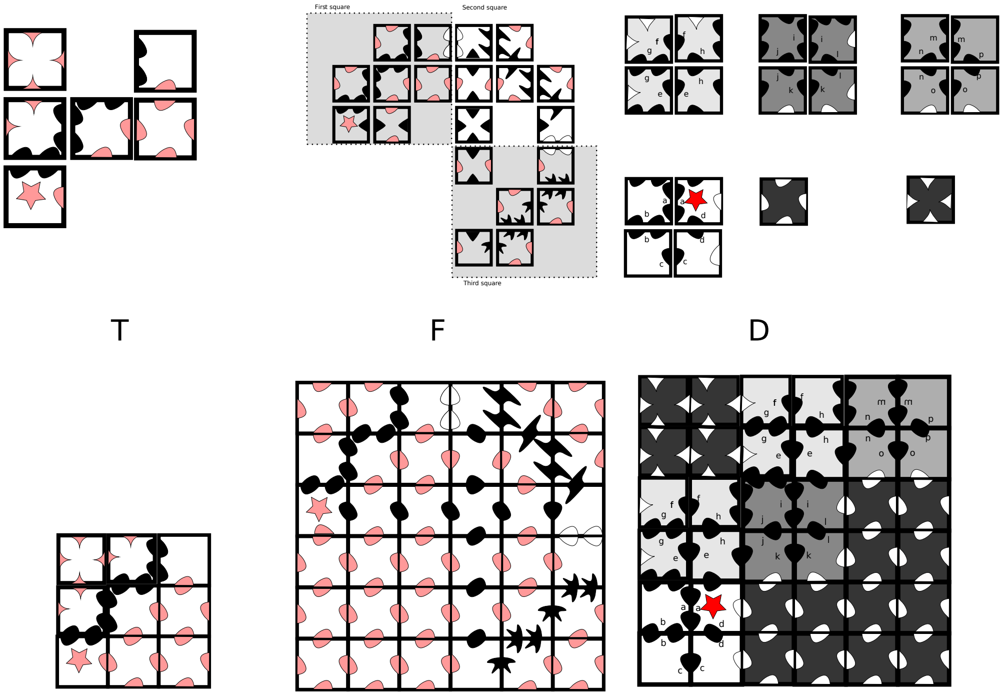


Figure 1: The three tilesets T,F and D, and an example final production for each of them.

The dynamic of T Our reference tileset T, taken from [9] assembles the set of all squares. That is, from the configuration $\{(0,0) \mapsto \mathit{seed}\}$, T eventually reaches a configuration whose shape is $[0,n] \times [0,n]$, where n is determined during the assembly.

The assembly can be decomposed in two concurrent processes: the construction of the diagonal, and the filling of the square. The filling of the square is conditioned by the construction of the diagonal: if the diagonal has been built up to the point (k,k), filling tiles can only be added in $[0,k] \times [0,k]$. This condition ensures that the shape we get when the diagonal process is stopped by a *stop* tile and the filling process is completed is a square.

The dynamic of F The tileset F assembles the set of all squares of even size. This set is exactly the image of L by an homothety of factor 2. The assembly process is the following : in a first phase, a square is built as by T, then another of the same size n, then a $n \times 2n$ rectangle. Thus, the final shape is a $2n \times 2n$ rectangle. This assembly process is not at all related to the assembly process of T and the fact that the final productions of F are the images of those of T is somewhat a “happy coincidence”.

The dynamic of D The set of shapes assembled by D is also the set of all squares of even size, but the way the assembly works resembles much more to T: there is also a diagonal being built, and completed into a square. More precisely, let us consider the set of all 2×2 patterns which appear in squares whose lower-left corner is in $\{2k, 2k' | k, k' \in \mathbb{Z}^2\}$. We call these patterns *macrotiles*. Each of these macrotiles can be identified to a tile of T, and the interactions between these macrotiles “look like” the interactions between the corresponding tiles of T.

If we take a (non-terminal) production p of T, and replace each tile with the matching macrotile, the result is a (non-terminal) production p_D of D. The addition of a tile in p corresponds to the addition of the four tiles of the macrotile in D, and the addition of a tile in p is possible if and only if it is possible to add the tiles of the macrotile in p_D.

We say that the dynamics of a tileset D is the n-scaling of that of T whenever one can find a set of macrotiles of size n such that the interactions between the macrotiles of D correspond to the interactions between the tiles of T, and any production of D can be cut in macrotiles.

2.2 Formal definitions

We now define more formally the notion of scaling the assembly of a tileset T. In this section, T, U are two tilesets, and the question is : "is the dynamic of U a scaling of that of T at scale s ?" We first define our transformations on shapes then extend them to patterns and dynamics.

Definition 4. *Let σ be a shape. Then $h_s(\sigma)$ is $\{(x,y)|(\lfloor x/s\rfloor, \lfloor y/s\rfloor) \in \sigma\}$.*

In order to be able to extend this definition to patterns, where each point is either empty or contains a tile, we need to know what to do with this information. For this, we will consider that $s \times s$ covered by tiles of U form "macrotiles".

Definition 5 (s-macrotile, interpretation)**.** *A* s-macrotile *on U is a function $[0, s[^2 \rightarrow U$. U_s is the set of s-macrotiles on U. For a pattern p, the macrotile at (sx, sy) in p is the function from $[0, s[^2$ to U defined by $(i, j) \mapsto p(xs + i, ys + j)$.*

An interpretation *i is a function from U_s to T.*

Now that we have grouping functions we can map patterns onto patterns. $i_s(p)$ is the pattern p shrinked by a factor s, and reinterpreted in T (when this definition makes sense).

Definition 6 (shrinking)**.** *Any pattern p on U such that there is a shape $\sigma \subset \mathbb{Z}^2$ such that $shape(p) = h_s(\sigma)$ is said to be* s-shrinkable*.*

Given an interpretation i, for a shrinkable pattern p, $i_s(p)$ is the pattern on T defined by:

- *$shape(i_s(p))$ is the σ from above,*
- *for all $(x, y) \in \sigma$, if m is the s-macrotile at (xs, ys) in p, then $(i_s(p))(x, y) = i(m)$.*
- *otherwise, $(i_s(p))(x, y)$ is undefined.*

$D_{s,U}$ is the sublattice formed by the shrinkable productions of U.

Definition 7. *Let D_U be the lattice of productions of U, and D_T be the lattice of productions of T. We say that the dynamics of U is the image of the dynamics of T at scale s when*

1. *There is an interpretation i mapping the s-macrotiles of U to T, such that i_s is an isomorphism between $D_{s,U}$ and D_T.*
2. *for every production c of U, there is a $c' \geq c$ which is shrinkable.*

This definition formalizes the similarity we had between T and D in the example: we could attribute to each 2-macro-tile of D a tile of T such that this macro-tile behaved like the associated tile. Thus, as we can expect, we get the following corollary:

Corollary 1. *Let U and T be two tilesets, L_T and L_U the set of shapes they assemble, if the dynamics of U is the image of the dynamics of T with the interpretation i, then $L_U = h_s(L_T)$*

Proof. Let p be a final production of T. $i_s^{-1}(p)$ is soundly defined since i_s is an isomorphism between D_U and D_T, it is a final production of U, and its shape satisfies $shape(i_s^{-1}(p)) = h_s(shape(p))$.

Let p be a final production of U, p is shrinkable, and $i_s(p)$ is a final production of T, and its shape satisfies $shape(p) = h_s(shape(i_s(p)))$. □ □

Our condition does not give a transition-to-transition matching between D_T and $D_{s,U}$, but only a path-to-path matching. This is because the parallelism in U can make several transitions from T take place at once.

Yet, it does allow us to link the transitions involved in both paths: to each path p through the dynamics of U, we can associate another path p' with the same transitions, from which one can extract a sequence of shrinkable patterns $(p*)$ such that the sequence $(i_s(p*))$ is a valid path in the dynamics of T. Let T and U be two tile-sets, s be an integer and i be an interpretation function of U_s in T, such that the dynamics of U is the image of that of T at scale s. Let $p = p(0) \dots p(n)$ be a path in the dynamics of T. There is a path $p' = i_s^{-1}(p(0)) \dots i_s^{-1}(p(1)) \dots i_s(p(n))$ in the dynamics of U. This path is not unique, and there are paths going from $i_s^{-1}p(0)$ to $i_s^{-1}(n)$ without going through all the $i_s^{-1}(p(k))$, but as they have the same extremities as p', they all have the same transitions, but in a different order.

3 Zooming and self-assembly

3.1 Zooming sometimes breaks dynamics

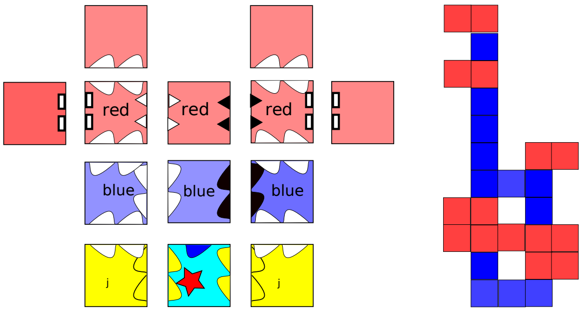
Having defined a notion of preservation of dynamics, we would like to see if there is a way to scale the dynamics of any tileset. The answer is that it is not possible without some more conditions. What does it mean? It means that the notion of locality which is used by these self-assembling tilings is very local, and thus cannot be scaled. There is then an intrinsic scale in the process. By putting some more conditions on how the assembly takes place, we can scale the dynamics.

To show that zooming (or scaling) necessarily breaks the dynamics of an assembly, we use the fact that in our model, locality means simultaneity in reactions. Thus, wherever we break locality, we will need synchronization, which means putting tiles in advance. This is what breaks the dynamics.

Theorem 1. *There is a tileset T such that there is no Ts whose dynamics is the image of that of T at scale s for $s \geq 2$.*

Proof. Let T be the tileset of figure 2(a). T behaves thus: if the seed is at $(0, 0)$, then when the assembly stops, there are two integers l_1 and l_{-1} such that there are tiles in the following places:

- in $\{-1\} \times \{0, \dots, l_{-1}\}$; these tiles can each be either red or blue
- in $\{1\} \times \{0, \dots, l_1\}$, also in red or blue,
- in the subset S of $\{0\} \times \{0, \dots, h_0\}$ defined by $\{(0, y) | \sigma_E(-1, y) = \sigma_W(1, y)\}$ (where $h_0 = min(l_1, l_{-1})$).
- In $\{-2\} \times \{0, \dots, l_{-1}\}$, wherever the tile in the same row at $x = -1$ is red



(a) The tiles

(b) The problematic part of its dynamics

Figure 2: our counter-example

- In $\{2\} \times \{0, \dots, l_1\}$, wherever the tile in the same row at $x = 1$ is red.

In the final productions, there are tiles in the central column whenever there are tiles at both $x = 2$ and $x = -2$ in the same row, or whenever there are neither.

This property cannot be ensured at greater scale. To show this, we do a proof *ad absurdum* by considering a system T_S which is a s-grouping of T. Let us consider the fragment of the dynamic of T represented on figure 2(b), in which no tile can be added at $(0, 1)$ or $(0, 2)$. This guarantee is given if the dependencies between tiles can be expressed by a poset.

Let $c'ij$ be the antecedent of cij by i_s, since i_s is an isomorphism, the $c'ij$ are in the same order as the cij. From $c'21$ and $c'23$, no tile can be added in the central column. Thus, there can be no strength 2 glue on the marked edges in $c'21$ and $c'23$ (represented in bold).

It is also impossible on the marked regions of $c'22$, because then they would have been added before either $c'11$ or $c'12$, and would be present in $c'21$ or c_s23. As there are no strength 2 glues on the marked areas, no tile can be added in the central column. This contradicts the fact that the dynamics of T_s is the image of that of T'. □ □

Here, T was unscalable because the locality of the tiles could not be scaled: sometimes, adding a macrotile to a configuration of Ts would have involved two concurrent processes, none of which would have got the whole information on the macrotile to be added. So, in order to scale dynamics, we need a guarantee that we will always be able to have all the information on the tile to be added at one place.

3.2 The order condition

To avoid this problem, one can add the classical Row-Column condition [4], or an extension, the *order condition*. This condition removes all effects such as shown in the previous section, and allows us to scale the dynamics. It states that one can associate, to each tile in a production, a direction which corresponds to the way it has extended the production when it was added. Thus, as we know where each tile will extend the productions, we can put the information on colors and glues where it is needed, which allows us to scale the tiling.

Definition 8 (Order condition)**.** *Let c be a configuration. If there is a poset $<_c$ on the shape of c*

such that the shapes of the productions $p < c$ are the ideals of $<_c$, then c is said to satisfy the order condition*, and $<_c$ is its* dependency order.

If all the productions of a tileset obey the order condition, we will say that it is an *ordered tileset.* Intuitively this condition states that the dependencies between the tiles can be presented as a poset, and that this poset depends only on the production and not on its history. $<_c$ represents the dependencies between tiles in c as a tile can be attached to a production p in order to give c if and only if all of its predecessors according to $<_c$ have already been attached. With this condition, the direction of each tile (as defined above) can be decided by looking only at $<_c$ and not at the history of the assembly. The direction of a tile is the opposite of the relative location of its predecessors for $<_c$. For example, if the predecessors of z are $S(z)$ and $W(z)$, the tile at z will have direction NE. This condition is slightly more general than Rothemund's RC condition[4], as it allows to build non-convex patterns. Yet, it is quite natural, and most if not all tilesets in the literature obey this order condition.

A corollary of this condition is that there is never "too much glue" when adding a tile: whenever a tile is added to a configuration, the sum of the glues on its adjacent sides is exactly 2

3.3 A Construction for the order case

Given this order condition, we are able to construct a scaled image of a given tile-set. We do this by cutting each of the tiles in s^2 pieces (where s is the scaling factor), and putting new glues on the new edges.

Theorem 2. *Let T be an ordered-tileset, and s an integer. Then there exists a Ts whose dynamics is the image of that of T at scale s. Furthermore, Ts is also ordered.*

Proof. To each tile of T, one can associate, thanks to the order condition, a set of directions in which the tile can extend the productions. For example, the set of directions associated with a tile having a glue of strength 2 on its southern edge and glues of strength 1 on the other edges is $\{N, SE, SW\}$.

For each tile t and each direction d in which t can extend productions, t^d is the tile t with arrows on its edges showing that t has been added in direction d. These arrows indicate whether an edge is an input or output edge: for each edge e, the arrow points into the tile (in direction $-e$) if the angle between e and d is acute (either 0 or 45 degrees) , and out of the tile otherwise (in direction e). A side with an arrow pointing into the tile is an input side, and a side with an arrow pointing out of the tile is an output side. See figure 3.3.

We consider the tileset T' whose color set is $\Sigma \times \{N, S, E, W\}$, and whose tiles are the t_d for $t \in T$ with strength function s_T. Clearly T' has the same dynamics as T, because of the order condition. We then build a tileset T_s having which is a scaling of T', and thus of T.

Ts is defined as $Seed \cup Replicas$.

$Replicas$ is a set of pieces of tiles in T'. Each tile in T' is cut into s^2 squares of size $1/s$, and the internal edges are given new colors, each unique to the position of the edge inside the tile and to the tile. The details of these glues are shown on figure 3.3. Glues of strength 2 are only present once per edge, and are replaced by equivalent force 1 glues elsewhere on the edge. By convention, they only appear in the uppermost and leftmost parts of each macrotile.

$Seed$ is a set of tile which gives a square of size s with on its edges, the color of the edges of the seed of T.

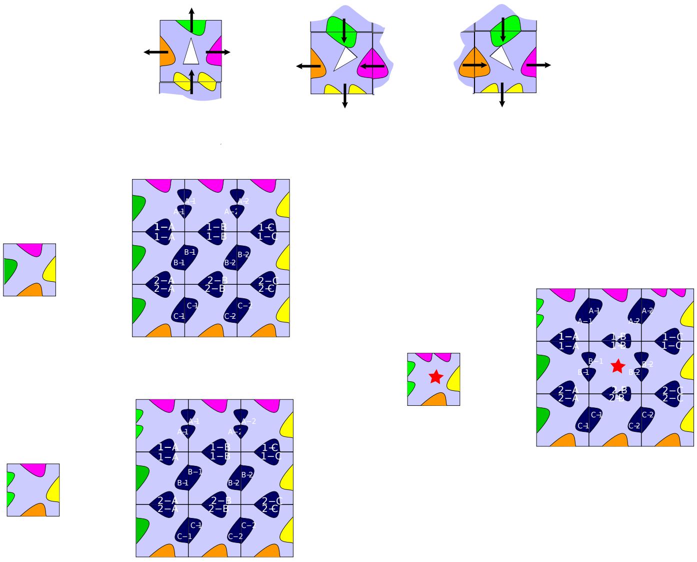


Figure 3: Top, adding the arrows to the tiles of T: the triangle is the direction of the tile; bottom, cutting into pieces for *Seed* and *Replicas*; all internal edges are unique within each macrotile and between macrotiles.

The interpretation i that we use for this construction is simple: the macrotiles which correspond to the definition of *Replicas* are mapped to the matching tile of T. The macrotile corresponding to the definition of *Seed* is mapped to the seed of T. The other macrotiles are not actual.

Let us now look at the assembly of this tile set. Let us show that the dynamics of Ts is the image of that of T'.

We use the notion of local production. A *local production* is a configuration which appears when building an actual macrotile m. A local production for m is a configuration included in the square $[0, s)^2$ which can be reached from the empty square with the input colors for m on the sides adjacent to its input sides.

We need to show that if $i_s(c) \leq i_s(c')$ are productions of T', then $c \leq c'$. To show this, it is sufficient to show that if there is a transition between $i_s(c)$ and $i_s(c')$, then $c \leq c'$. This is easily seen, one only needs to reproduce the constructions of figure 3.3, which is always possible.

We add an element $\perp$ at the bottom of $D_{T'}$ and D_{T_s} (that is, for all $c \in D_{T'}, c \geq \perp$). We will say that $\perp$ is shrinkable, and $i_s(\perp) = \perp$. We prove the following lemma:

Lemma 1. *For any shrinkable production d and any $c \geq d$, there is a configuration e of T' such that:*

- *$shape(e)$ is the smallest domain $\sigma \subset \mathbb{Z}^2$ such that $h_s(\sigma) \supset shape(c)$)*
- *$e \geq i_s(d)$*
- *For any square $S = [xs, (x+1)s[\times[ys, (y+1)s[$, c restricted to S is a local production of the macro-tile corresponding to $e(x, y)$*
- *and c verifies the order condition*

Proof. For any d, we prove the lemma by induction on c. When $c = d$, we take $e = i_s(d)$.

Let $c \geq d$, and τ be a transition from c to c' where one adds a tile t at $(sx+i, sy+j), 0 \leq i, j < s$. Let e be given by the induction hypothesis for c.

If $(x, y) \in \mathit{shape}(e)$, then e also verifies the lemma for c'. The condition on the shapes is clearly true, and also that $e \geq i_s(d)$. Since c is made of local productions and obeys the order condition, t can only be placed if it has its neighbors on its input sides (because of the arrows on the sides). Thus, it can only be added in accordance with the local productions, and cannot break the order condition.

If $(x, y) \notin \mathit{shape}(e)$, then let $t' \in T$ be the tile corresponding to the macrotile in which t appears. Let t_1 and (if needed) t_2 be the neighbors of t on its input sides, and the corresponding $t'_1, t'_2 \in T$. As t could be added next to t_1 and t_2, t' can be added between t'_1 and t'_2. Let $e' = e \cup \{(x, y) \to t'\}$. $e' \geq i_s(d)$, and it has the smallest shape such that $\mathit{shape}(c') \subset h_s(\mathit{shape}(e'))$. t is a local production of the macrotile corresponding to t', and c' has an order of dependency. ☐

With this lemma, we get that if $c \leq c'$ and c, c' are shrinkable, then $i_s(c) \leq i_s(c')$, as when c is shrinkable, $i_s(e) = c$.

Let c be a production of Ts, then there is a corresponding production e of T', such that c is made of local productions of e. Let $(x_0, y_0, t_0) \ldots (x_k, y_k, t_k)$ be a chain of transitions from $\perp$ to e, and i be the smallest k such that $\mathit{shape}(c) \cap ([sx_k, s(x_k+1)[\times[sy_k, s(y_k+1)[) \neq [sx_k, s(x_k+1)[\times[sy_k, s(y_k+1)[$, if it exists. Then, as $\mathit{shape}(c) \cap ([sx_k, s(x_k+1)[\times[sy_k, s(y_k+1)[)$ is a local production and its input neighbor(s) are complete, one can add a tile to c in $[sx_k, s(x_k+1)[\times[sy_k, s(y_k+1)[$. So if c is not shrinkable, there is a $c' > c$ shrinkable.

So the dynamics of Ts is the image of that of T'. ☐

Conclusion

We have defined a notion of transformation of a tile-set that allows us to compute geometrical transformations of the plane within the tile-set, in parallel to whatever construction the tile-set does. This notion allows to pinpoint which aspects of locality a given transformation can break. We gave a demonstration framework for these questions. The fact that in all generality, this notion is not compatible with zooming can be interpreted as an acute sensibility of some self-assembly processes with locality.

Looking at dynamics for making transformation gives a new kind of algorithms on assemblies. Further investigation could lead to more uniform transformations (less dependence on the tileset): if we accept to introduce some inaccuracy in the tileset, then it is possible to have a set G of "growing tiles" such that for a tileset T, the dynamics of $G \cup T$ is the image of that of T by an homothety.

This construction can also be seen as a framework for implementing other geometrical transformations in self-assembly. The macrotiles construction is in fact a kind of tensor products between two tilesets, one assembling a finite pattern, and the other the image of the grid by a transformation. Whenever we have these two elements[2], we can implement the transformation. This expands in a very interesting manner the toolbox of the self-assemblist, allowing for new decompositions of shapes to be built.

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[2] the grid part is not trivial for more involved transformations

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A the lattice structure of the dynamics

Theorem 3. *Let T be a tileset, and p a production of T, then the set $\{\{seed\} < x < p\}$ is a lattice.*

Proof. This set is a po-set, we need to define the upper and lower bound of two elements. Let $\{seed\} < c, c' < p$.

We first define $c \vee c'$. As $c < p$ and $c' < p$, c and c' agree wherever they are defined. Let d be the configuration defined on $shape(c) \cup shape(c')$ by $d(x,y) = c(x,y)$ for $(x,y) \in shape(c)$ and $d(x,y) = c'(x,y)$ for $(x,y) \in shape(c')$. Any configuration greater than c and c' is greater than d. Let us show that $d > c$. We consider a maximal element m of the set $\{c \leq x | shape(x) \subset shape(d)\}$. If $m \neq d$, then it is possible to add a tile to m restricted to $shape(c')$, so it is also possible to add a tile to m in $shape(d)$, which contradicts the maximality of m. Thus, $c < d$. So $d = c \vee c'$.

To define $c \wedge c'$, we consider the set $S = \{x | x < c \wedge x < c'\}$. $S \subset \{x | x < c\}$, so it has an upper bound, which is in fact a maximum. This maximum is $c \wedge c'$. □

In the construction of $\wedge$, we didn't use the fact that $c, c' < p$, so we have the following theorem:

Theorem 4. *Let T be a tileset, the set of its production is a $\wedge$-semi-lattice.*