[1] Let $\mathcal{R}$ denote the space of lattices in $\mathbb{C}$, that is $\mathcal{R} = \mathrm{GL}_2^+(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. By discreteness of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{GL}_2^+(\mathbb{R})$, this is a locally compact (Hausdorff) space.

The group $\mathbb{C}^*$ acts (continuously and) properly by multiplication on $\mathcal{R}$, with generic stabilizer $\{\pm 1\}$, and two orbits with (cyclic) stabilizers of orders 4 and 6, namely those of $R = \mathbb{Z} + i\mathbb{Z}$ and $R = \mathbb{Z} + \rho\mathbb{Z}$ ($\rho = e^{2i\pi/3}$). Note that -1 acts trivially on $\mathcal{R}$, and that the induced action of $\mathbb{R}_+^*$ on $\mathcal{R}$ is free, with $\mathcal{R}$ homeomorphic to $\mathcal{R}_1 \times \mathbb{R}_+^*$, where $\mathcal{R}_1 \subset \mathcal{R}$ is the subspace of lattices of area 1.

Recall that for a lattice R in $\mathbb{C}$, the complex torus $T = \mathbb{C}/R$ is Riemann surface whose isomorphism class is given by the orbit of R under multiplication by elements of $\mathbb{C}^*$. The quotient space $\mathcal{M} = \mathbb{C}^* \backslash \mathcal{R} = \mathbb{C}^* \backslash \mathrm{GL}_2^+(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ is thus the "moduli space" of (one dimensional) complex tori. Moreover, the punctured curve $T' = (\mathbb{C}/R) \setminus \{0\}$ is isomorphic to the complex plane curve defined by *Weierstrass equation*

$$y^2 = 4x^3 - g_2(R)x - g_3(R)\ , \quad (x,y) \in \mathbb{C}^2$$

with

$$g_2(R) = 60 \sum_{\lambda \in R\setminus\{0\}} \frac{1}{\lambda^4}\ , \quad g_3(R) = 140 \sum_{\lambda \in R\setminus\{0\}} \frac{1}{\lambda^6}\ ,$$

and the isomorphism is given by $z + R \mapsto (\wp(z), \wp'(z))$ for the Weierstrass function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in R\setminus\{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)\ .$$

Any orbit of $\mathbb{C}^*$ in $\mathcal{R}$ contains a lattice of the form $R = \mathbb{Z}\tau + \mathbb{Z}$ for τ in the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, and the different values of τ giving a lattice in the same orbit of $\mathbb{C}^*$ constitute the orbits of the group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathcal{H}$ by $\tau \mapsto (a\tau + b)/(c\tau + d)$, $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$. Indeed if $\lambda(\mathbb{Z}\tau + \mathbb{Z}) = \mathbb{Z}\tau' + \mathbb{Z}$ for some $\lambda \in \mathbb{C}^*$, one has $\tau' = \lambda(a\tau + b)$, $1 = \lambda(c\tau + d)$ for $a, b, c, d \in \mathbb{Z}$ $ad - bc = \pm 1$ hence by quotienting $\tau' = (a\tau + b)/(c\tau + d)$ and necessarily $ad - bc = 1$ since τ, τ' are in $\mathcal{H}$. In particular $\mathbb{C}^* \backslash \mathcal{R} \simeq \mathcal{H}/\Gamma$.

The functions g_2, g_3 on $\mathcal{R}$ obviously satisfy $g_k(cR) = c^{-2k} g_k(R)$, $c \in \mathbb{C}^*$, and the discriminant of the cubic polynomial $P(x)$ on the right of Weierstrass equation is given (up to a multiplicative constant) by $\Delta = g_2^3 - 27g_3^2$, which satisfies $\Delta(cR) = c^{-12}\Delta(R)$.

The discriminant Δ doesn't vanish at any $R = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \in \mathcal{R}$ since Weierstrass function $\wp : \mathbb{C}/R \to \hat{\mathbb{C}}$, being of degree only 2, necessarily takes distinct values at its critical points, which are the three zeros $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2 \pmod{R}$ of its derivative $\wp'$ (clearly an odd meromorphic function on $\mathbb{C}/R$ of degree 3). Those values being the roots of $P(x)$, one obtains $\Delta(R) \neq 0$.

The map

$$\Phi : \mathcal{R} \to \mathbb{C}^2\ , \quad R \mapsto \left(\frac{g_2(R)}{3}, g_3(R) \right)$$

thus takes its values in the complement of the curve Σ defined by $y^2 = x^3$ and intertwines the actions of $\mathbb{C}^*$ on $\mathcal{R}$ and $\mathbb{C}^2$ given for $c \in \mathbb{C}^*$ by $R \mapsto cR$ and $(x, y) \mapsto (c^{-4}x, c^{-6}y)$.

We will prove

Theorem. *Φ is an homeomorphism $\mathcal{R} \to \mathbb{C}^2 \setminus \Sigma$.*

1. These complements were inspired in part by Hubbard and Pourezza text "Ths pace of closed subgroups of $\mathbb{R}^2$" `www.math.cornell.edu/~hubbard/ClosedSubgroupsR2.pdf`

Quotienting on both sides by the action of the subgroup $\mathbb{R}_+^* \subset \mathbb{C}^*$, this will give an homeomorphism between the space $\mathcal{R}_1 \simeq \mathcal{R}/\mathbb{R}_+^*$ of lattices of area 1 in $\mathbb{C}$ and the quotient $(\mathbb{C}^2\backslash\Sigma)/\mathbb{R}_+^*$ where the action of $t \in \mathbb{R}_+^*$ is $(x, y) \mapsto (t^2x, t^3y)$ (which corresponds via Φ to the action $R \mapsto t^{-1/2}R$ on $\mathcal{R}$).

But this last quotient is homeomorphic to the complement of the $(2, 3)$ torus knot (a.k.a. trefoil knot) in $\mathbb{S}^3$. Indeed all curves $t \mapsto (t^2x, t^3y)$, $t > 0$, $(x, y) \in \mathbb{C}^2 \setminus \{0\}$ intersect transversely once the unit sphere $\mathbb{S}^3$ and on this sphere the circle $\mathbb{S}^1$ acts via $(x, y) \mapsto (e^{2i\theta}x, e^{3i\theta}y)$, $e^{i\theta} \in \mathbb{S}^1$. The only non-free orbits are $\mathbb{S}^1 \times 0$ and $0 \times \mathbb{S}^1$, with stabilizers of orders 3 and 2 respectively, and the (free) orbit of the point $(r, r^{3/2})$ with $r^2 + r^3 = 1$, $r > 0$ is the trefoil knot. Thus we deduce

Corollary. *The space $\mathcal{R}_1$ of lattices of unit area in $\mathbb{C}$ is homeomorphic to the complement of the trefoil knot in $\mathbb{S}^3$.*

To show that Φ is a homeomorphism, observe first that it is injective. Indeed if $g_2(R) = g_2(R')$, $g_3(R) = g_3(R')$, the Weierstrass curves are the same, hence the two punctured tori also, implying $R' = cR$ because the unpunctured tori are then isomorphic (by the removable singularity theorem). If g_2 and g_3 are non-zero this implies $c^{-4} = c^{-6} = 1$, thus $c^2 = 1$ and $R = R'$. If $g_2 = 0$, $g_3 \neq 0$, $c^6 = 1$ and the curve $y^2 = 4x^3 - g_3$ has an automorphism $(x, y) \mapsto (\rho x, -y)$ of order 6, hence $\mathbb{C}/R$ also, implying $R' = cR = R$. If $g_3 = 0$, $g_2 \neq 0$, $c^4 = 1$ and the curve $y^2 = 4x^3 - g_2x$ has an automorphism $(x, y) \mapsto (-x, iy)$ of order 4 hence $R' = cR = R$.

Next, to see that Φ is *locally* an homeomorphism [2], one can resort to Brouwer's invariance of domain, but since the map is so explicit a simpler argument should be possible, for instance trying to use the inverse function theorem.

Observe that, using the action of $\mathbb{C}^*$ on $\mathcal{R}$ and on $\mathbb{C}^2 \setminus \Sigma$, one can work near a lattice $R_0 = \mathbb{Z} + \mathbb{Z}\tau_0$, $\tau_0 \in \mathcal{H}$, and parametrize locally $\mathcal{R}$ near R_0 by $R = R(\lambda, \tau) = \lambda(\mathbb{Z} + \mathbb{Z}\tau)$ with $\lambda \in \mathbb{C}^*$ near 1 and $\tau \in \mathcal{H}$ near τ_0 (λ and $\lambda\tau$ will be the only points of R near 1 and τ_0 respectively if R is near enough R_0).

Then
$$(\lambda, \tau) \mapsto \Psi(\lambda, \tau) = \Phi(R(\lambda, \tau)) = (\lambda^{-4}g_2(\tau)/3, \lambda^{-6}g_3(\tau))$$
is holomorphic in (λ, τ), and is injective near $(1, \tau_0)$. It is then known that the differential $D\Psi(1, \tau_0)$ is injective [3], which would suffice, but a direct elementary argument is the following. Since $\partial_\lambda\Psi(1, \tau_0) \neq (0, 0)$, it is easy to find a linear change of coordinates (L_1, L_2) on $\mathbb{C}^2$ such that the derivative at 1 of $\lambda \mapsto \Psi_1(\lambda, \tau_0) = L_1(\Psi(\lambda, \tau_0))$ is non-zero. By the inverse function theorem,
$$(\lambda, \tau) \mapsto (s, \tau) = (\Psi_1(\lambda, \tau), \tau)$$
is a biholomorphism between neighbourhoods of $(1, \tau_0)$, (s_0, τ_0) and (with hopefully obvious notations)
$$(s, \tau) \mapsto \tilde{\Psi}(s, \tau) = (s, \tilde{\Psi}_2(s, \tau)) = \Big(s, L_2(\tilde{\Psi}(s, \tau))\Big)$$
is holomorphic and injective near (s_0, τ_0). But this forces $\partial_\tau\tilde{\Psi}_2(s_0, \tau_0)$ to be non-zero (by injectivity the *one variable* map $\tau \mapsto \tilde{\Psi}_2(s_0, \tau)$ near τ_0), hence $\tilde{\Psi}$ is a biholomorphism near (s_0, τ_0). Unwinding the coordinate changes, one conclude that Ψ is a local biholomorphism near $(1, \tau_0)$, in particular a local homeomorphism.

2. equivalently an open map, but this seems not easier to prove.
3. see e.g. `https://math.stackexchange.com/a/497551/25917`

To show that Φ is a homeomorphism, it is now enough to show that it is proper, because an injective proper local homeomorphism between connected locally compact spaces is an homeomorphism.

We will show that the map $J : \mathcal{H}/\Gamma \to \mathbb{C}$ defined by [4]

$$J(\tau) = 12^3\, g_3(\tau)^2/\Delta(\tau) = 1728\, g_2(\tau)^3/(g_2(\tau)^3 - 27 g_3(\tau)^2)$$

is proper (and in fact an homeomorphism). It is then not hard to conclude that Φ is proper, noting that J is the composition of Φ with $(x, y) \mapsto 1728/(1 - y^2/x^3)$.

Using a previous exercise sheet, one finds that with $q = \exp(2i\pi\tau)$, $\tau \in \mathcal{H}$,

$$g_2(\tau) = \frac{4}{3}\pi^4 \left(1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n\right)$$

$$g_3(\tau) = \frac{8}{27}\pi^6 \left(1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n\right)$$

which with some more calculations leads to

$$J(\tau) = \frac{1}{q} + O(1).$$

In particular the function J on $\mathcal{H}/\Gamma$ tends to ∞ for $q \to 0$, corresponding to $\mathrm{Im}(\tau) \to +\infty$, i.e. going to infinity in $\mathcal{H}/\Gamma$. Thus J is proper. Moreover, J defines an isomorphism $\mathcal{H}/\Gamma \to \mathbb{C}$, since it has a simple pole for $q \to 0$, implying both that q is a (holomorphic) coordinate in the neighbourhood of infinity in $\mathcal{H}/\Gamma$ and that J is a global coordinate.

One can also exhibit a continous inverse map to Φ, namely the "period mapping" $P : \mathbb{C}^2 \setminus \Sigma \to \mathcal{R}$. For $(a, b) \in \mathbb{C}^2 \setminus \Sigma$, define $P(a, b)$ as the set of values of the integral $\int_\gamma \omega_{a,b}$, where $\omega_{a,b} = dx/y$ is a holomorphic one-form on the compactification $C_{a,b}$ of the affine curve with equation $y^2 = 4x^3 - ax - b$ and $\gamma : \mathbb{S}^1 \to C_{a,b}$ is a closed piecewise differentiable curve. This is a subgroup of $\mathbb{C}$, as one sees by considering change of orientation on γ to obtain the opposite and joining two curves by a path and its reverse to obtain the sum. That it is at most of rank 2 comes from the fact that two homotopic (or even only homologous) curves γ give (by Cauchy) the same value of the integral and that $\pi_1(C_{a,b}) = \mathbb{Z}^2$.

But that it is discrete and of rank precisely 2 is much more subtle and results from so-called *Riemann bilinear relations* (in this genus 1 case).

If $x_1, x_2, x_3 = -x_1 - x_2 \in \mathbb{C}$ are the zeros of $4x^3 - ax - b$ (distinct since $a^3 - 27b^2 \neq 0$), $P(a, b)$ is generated for instance by $\omega_1 = 2\int_{x_1}^{\infty} dx/y$, $\omega_2 = 2\int_{x_1}^{x_2} dx/y$ (for some "reasonable" choice of paths of integration, e.g. straight). Then one can show that $\mathrm{Im}(\overline{\omega_1}\omega_2) \neq 0$, giving rank 2 and discreteness.

4. the coefficient 1728 is a traditional normalization, natural only from later developments (and not only to have a pole with residue 1). It obviously could have been omitted here.