Exercise 1

a) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on an open subset Ω of $\mathbb{C}$. Show that if f is not constant near $a \in \Omega$, the image of any neighborhood of a is a neighborhood of $f(a)$ (f is *open* at a).

b) Deduce that if the function $|f| : \Omega \to \mathbb{R}$ reaches its supremum at a point a of Ω, f is constant in the connected component of a in Ω ("maximum modulus principle").

c) Same for $\mathrm{Re}(f) : \Omega \to \mathbb{R}$ ("maximum principle").

d) Conclude that on a compact *connected* Riemann surface X, all holomorphic functions $f : X \to \mathbb{C}$ are constant. Hence

$$\mathcal{O}_X(X) = \mathbb{C} \ .$$

This applies to $X = P^1(\mathbb{C})$.

Exercise 2 Recall, or take as definition, that a *meromorphic function* $f : X \dashrightarrow \mathbb{C}$ on a Riemann surface X is a holomorphic function $X \setminus S \to \mathbb{C}$ for some closed discrete[1] subset S of X such that f has at most a pole at each point $p \in S$.

a) Show that this is the same as a holomorphic map $f : X \to P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, not identically ∞ on any connected component of X.

b) Explain why any rational fraction $F \in \mathbb{C}(T)$ defines a meromorphic function f on $P^1(\mathbb{C})$, which conversely determines F (and is usually identified to it). The goal of the four questions to follow is to show that all meromorphic functions on $P^1(\mathbb{C})$ are of this type.

c) Let $f : P^1(\mathbb{C}) \dashrightarrow \mathbb{C}$ be a non-constant meromorphic function on the Riemann sphere. If $a_1, \ldots, a_k$ are the poles of f in $\mathbb{C} = P^1(\mathbb{C}) \setminus \{\infty\}$, of respective orders $n_1, \ldots, n_k \in \mathbb{N}^*$, show that $g(z) = (z - a_1)^{n_1} \ldots (z - a_k)^{n_k} f(z)$ defines a meromorphic function g on $P^1(\mathbb{C})$, without poles in $\mathbb{C}$. Hence $g : \mathbb{C} \to \mathbb{C}$ is holomorphic.

d) If ∞ is a pole of order m of g, show that $|g(z)| \leq C|z|^m$ for $z \in \mathbb{C}$, $|z| \geq 1$ and some constant C.

e) Using Cauchy formula

$$g^{(k)}(0) = \frac{k!}{2i\pi} \int_{|z|=R} \frac{g(z)}{z^{k+1}}\, dz$$

deduce that g is a polynomial function of degree at most m on $\mathbb{C}$.

f) Conclude that any meromorphic function on $P^1(\mathbb{C})$ is given by a rational fraction $F \in \mathbb{C}(T)$.

g) Let $F \in \mathbb{C}(T)$ be a rational fraction, uniquely expressed as $F = P/Q$ for polynomials $P, Q \in \mathbb{C}[T]$ without common zero, with $Q \neq 0$ and unitary. Show that for all $t \in P^1(\mathbb{C})$ except a finite number, $f^{-1}(t)$ has cardinal $d = \max(\deg(P), \deg(Q))$.

h) Deduce that the automorphisms (biholomorphisms) of the Riemann surface $P^1(\mathbb{C})$ are given by the homographies $f(T) = (aT + b)/(cT + d) \in \mathbb{C}(T)$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ ("Möbius transformations"). They constitute the *projective linear group* of invertible complex 2 by 2 matrices up to scalars

$$\mathrm{PGL}(2, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* \simeq \mathrm{Aut}(P^1(\mathbb{C})) \ .$$

1. in particular finite if X is compact

$$\mathbb{C}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad z \mapsto \frac{az+b}{cz+d} \ .$$

i) Show that for any three disctinct points $a, b, c \in P^1(\mathbb{C})$, there is a unique $f \in \mathrm{PGL}(2, \mathbb{C})$ sending respectively a, b, c to $0, 1, \infty$ (or any other three distinct points).

j) Identify in $\mathrm{PGL}(2, \mathbb{C})$ the subgroups preserving the subsets $\mathbb{C} = P^1(\mathbb{C}) \setminus \{\infty\}$, $P^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, $H = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$, $\mathbb{S}^1 = \{z \in \mathbb{C}; |z| < 1\}$, $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ (an homography sending H onto $\mathbb{D}$ can be used).

Exercise 3 Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$.

a) Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map such that $f(0) = 0$. Show that one can write $f(z) = zg(z)$, for a holomorphic map $g : \mathbb{D} \to \mathbb{C}$. What is $g(0)$?

b) Using the maximum principle, deduce the *Schwarz lemma* : $|f'(0)| \leq 1$, and in case of equality, $f(z) = \lambda z$ for a complex number λ of modulus 1.

c) Let a be a point of $\mathbb{D}$. Verify that $z \mapsto (z - a)/(1 - \bar{a}z)$ is a biholomorphism $\mathbb{D} \to \mathbb{D}$.

d) Deduce that any biholomorphism $\phi : \mathbb{D} \to \mathbb{D}$ is of the form $\phi(z) = \lambda(z-a)/(1-\bar{a}z)$, for $|\lambda| = 1$, $a \in \mathbb{D}$, hence is the restriction to $\mathbb{D}$ of an automorphism of $P^1(\mathbb{C})$.

e) What about biholomorphisms of $H = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$?

f) And those of $\mathbb{C}$?

Exercise 4 Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$, with coordinates (u, t).

a) Express the stereographic projection from the "north pole" $\{(0, 1)\}$ to the horizontal (complex) plane $t = 0$

$$\phi_0 : U_0 = \mathbb{S}^2 \setminus \{(0, 1)\} \longrightarrow \mathbb{C} \ .$$

Same for $\phi_1 : U_1 = \mathbb{S}^2 \setminus \{(0, -1)\} \to \mathbb{C}$.

b) Verify that in $U_0 \cap U_1$, the product $\phi_0 . \overline{\phi_1}$ is constant equal to 1.

c) Deduce the domain, range and expression of the transition map $\overline{\phi_1} \circ \phi_0^{-1}$.

d) On each tangent plane of $\mathbb{S}^2$, the ambient euclidean metric of $\mathbb{C} \times \mathbb{R}$ induces a positive definite quadratic form η. This gives the standard riemannian metric of $\mathbb{S}^2$. Compare it with the other riemannian metric on the complement of the north pole obtained by "pull-back" from the standard metric $|dz|^2$ of $\mathbb{C}$ and the differential of ϕ_0.

e) Deduce the expression of the image of the spherical riemannian metric on $\mathbb{C} = \phi_0(U_0)$ (it should be of the form $a(z)|dz|^2$, showing that ϕ_0 is a *conformal* map from $U_0 \subset \mathbb{S}^2$ to $\mathbb{C}$.).