Exercise 1

a) Let $f : X \to Y$ be a *nonconstant* holomorphic map between connected Riemann surfaces. Show that for any point $p \in X$, denoting $q = f(p) \in Y$, there exist local holomorphic coordinates z,w centered at p and q respectively [1] and an integer $m \geq 1$ such that in these coordinates f is qiven by $w = z^m$. The integer $m = m(p) = m(f, p)$ doesn't depend on coordinates, and might be called the "multiplicity" of f at p. Show that the set of points [2] p with $m(f, p) > 1$ is a closed discrete subset of X, hence finite if X is compact.

b) Let $f : X \to Y$ be a bijective holomorphic map between Riemann surfaces. Show that it is a biholomorphism, *i.e.* that f^{-1} is holomorphic. What can one conclude if f is only injective ?

c) Let $f : X \to Y$ be a nonconstant holomorphic map between compact connected Riemann surfaces. Show that for any point q in Y, the set $f^{-1}(q)$ is finite, and that the sum of multiplicities $\sum_{p \in f^{-1}(q)} m(f, p)$ doesn't depend on q.

Exercise 2 Recall the "principle of removable singularities" : if U is open in $\mathbb{C}$ and $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, bounded in the neighbourhood of $a \in U$, it extends as a (continuous and) holomorphic function on U.

a) Let $f : \mathbb{C} \to \mathbb{C}$ be a *proper* holomorphic map, meaning that inverse images of compact subsets are compact, or equivalently that its extension to one point compactifications $\hat{f} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by $\infty \mapsto \infty$ is continuous. Show that ∞ is a pole of $\hat{f}$.

b) Conclude that f is a polynomial. Did you use Cauchy's formula ?

c) Show that each biholomorphism $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$.

Exercise 3 The goal of this (somewhat hard) exercise is to follow the proof by Gauss of the local uniformization theorem for real-analytic metrics (or conformal structures). In a first try, you can admit that real-analytic functions are "very well-behaved" : they add, multiply, compose (like differentiable functions). And that complex analytic functions of several variables satisfy "standard theorems" of real variable differentiable calculus : local inversion theorem, local existence of solutions of ordinary differential equations (with complex time !), for instance.

Let V be a neighbourhood (abbreviated nbhd in the sequel) of the origin in $\mathbb{R}^2$ and a, b, c three *real-analytic* functions from V to $\mathbb{R}$, meaning that their (real) Taylor series at each point of V converge in a nbhd of this point [3]. In particular a, b, c extend to a nbhd of $(0, 0)$ in $\mathbb{C}^2$ as holomorphic functions of two *complex* variables.

Assume that a, c and $ac - b^2$ are > 0 in V, *i.e.* that $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite.

1. meaning $z(p) = 0$, $w(q) = 0$
2. called "critical points" of f
3. for some C, M, and all j, k, $|\partial_x^j \partial_y^k f| \leq C j! k! M^{j+k}$

a) Show that there exists a real-analytic function $\lambda : V \to \mathbb{C} \setminus \mathbb{R}$ such that

$$a\lambda^2 + 2b\lambda + c = 0$$

in V. One can assume $\mathrm{Im}(\lambda) < 0$. The imaginary directions (complex lines in $\mathbb{C}^2$)

$$dx - \lambda(x,y)dy = 0,$$

$$dx - \overline{\lambda}(x,y)dy = 0$$

are the isotropic directions of the (positive definite) quadratic form

$$\begin{aligned} \eta_{(x,y)} &= a(x,y)\, dx^2 + 2b(x,y)\, dx\, dy + c(x,y)\, dy^2 \\ &= a(x,y)\, (dx - \lambda(x,y)dy)\, (dx - \overline{\lambda}(x,y)dy) \end{aligned}$$

which for (x,y) varying in the nbhd V of $(0,0)$ in $\mathbb{R}^2$ defines a (real analytic) *riemannian metric* there.

b) Show that the solution $\phi(x,t) = \phi_t(x)$ of the differential equation $\frac{d}{dt}\phi_t(x) = \lambda(\phi_t(x), t)$, with initial condition $\phi_0(x) = x$ defines a complex-analytic function of (x,t) in a neigbourhood of $(0,0) \in \mathbb{C}^2$.

c) Show that $(x,t) \mapsto (\phi_t(x), t)$ is a complex-analytic diffeomorphism on a nbhd of $(0,0) \in \mathbb{C}^2$, which fixes $(0,0)$.

d) Show that $(x,y) \mapsto w(x,y) = (\phi_y)^{-1}(x)$ is a complex-analytic function on a nbhd of $(0,0) \in \mathbb{C}^2$, with $w(x,0) = x$ and $\partial_y w(x,0) = -\lambda(x,0)$. More generally $\partial_y w(x,y) = -\lambda(x,y)\partial_x w(x,y)$ on a nbhd of $(0,0) \in \mathbb{C}^2$.

e) Deduce that w defines a diffeomorphism from a nbhd of $(0,0)$ in $\mathbb{R}^2$ to a nbhd of 0 in $\mathbb{C}$. It is thus a complex coordinate on a nbhd of $(0,0)$, which verifies $dw = \partial_x w(x,y)\,(dx - \lambda(x,y)dy)$. The choice of $\mathrm{Im}(\lambda) < 0$ ensures that it respects the standard orientations.

f) Conclude that in the complex coordinate $w = u + iv$ (on a nbhd of $(0,0) \in \mathbb{R}^2$), the metric η can be written $\eta = \rho|dw|^2 = \rho(du^2 + dv^2)$, where $u = \mathrm{Re}(w)$, $v = \mathrm{Im}(w)$, and ρ is a positive function. The coordinates (u,v) are called *conformal* (or *isothermal*) for the metric η.

g) Explain why for any other local complex coordinate w_1 satisfying the same conditions, the transition map $w_1 \circ w^{-1}$ is holomorphic.