Exercise 1 Let X be a connected Riemann surface.

a) Show that if $P \in X$, $\mathrm{ord}_P : \mathcal{M}(X) \to \mathbb{Z} \cup \{+\infty\}$ is a (discrete) valuation on the field $\mathcal{M}(X)$, *i. e.*

$$\begin{array}{rcl} \mathrm{ord}_P(0) & = & +\infty \\ \mathrm{ord}_P(f) & < & +\infty \quad \text{if } f \neq 0 \\ \mathrm{ord}_P(f+g) & \geq & \min(\mathrm{ord}_P(f), \mathrm{ord}_P(g)) \\ \mathrm{ord}_P(fg) & = & \mathrm{ord}_P(f) + \mathrm{ord}_P(g) \end{array}$$

Is it obvious that this valuation is non-trivial, meaning its image is not reduced to $\{0, +\infty\}$? What other examples of valuations do you know ?

b) If $f : X \to \mathbb{C} \cup \{\infty\}$ is a non-constant meromorphic function on X and $P \in X$, show that[1] $e_P(f) = \mathrm{ord}_P(f - f(P))$ if P is not a pole of f, and $e_P(f) = \mathrm{ord}_P(1/f) = -\mathrm{ord}_P(f)$ if P is a pole of f.

c) If $f : X \to Y$ is a *proper* holomorphic map, show that the branching set[2] $B_f \subset Y$ is closed and discrete.

Exercise 2 Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic with only simple zeros.

a) Show that $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = f(x)\}$ is a (non-compact) Riemann surface. What happens if f has multiple zeros ?

b) Show that $p : X \to \mathbb{C}$, defined by $p(x, y) = x$, is holomorphic, with branching set equal to $f^{-1}(0)$.

c) Assuming f is a polynomial of degree m, describe $p^{-1}(\mathbb{C} \setminus B(0, R))$ for large R, and the restriction of p to it.

Exercise 3 Let $f : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ be given by a non-constant rational fraction $f(z) = P(z)/Q(z)$, for complex polynomials P, Q without common zero. Let $d = \max(\deg P, \deg Q)$ be the "degree" of f.

a) Assuming that f has only simple poles (more precisely that $\infty \notin B_f$), show that $\sum_{P \in P^1(\mathbb{C})} (e_P(f) - 1) = 2(d-1)$. This is the "total number of ramification points of f, counted with multiplicity".

b) Deduce the same equality with no assumption on the poles of f (hint : use automorphisms).

Exercise 4 A *conformal structure* on a real vector space V is the datum of a positive definite quadratic form q (a euclidean structure) on V, up to multiplication by a scalar.

a) Show that a conformal structure on V is determined by the relation of orthogonality between vectors.

b) If V is two dimensional and oriented, show that a conformal structure on V is the same as an endomorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$ and (v, Jv) is a direct basis of V

1. the ramification index $e_P(f)$ of f at P is the same as the "local multiplicity" $m(f, P)$ of a previous sheet.
2. a.k.a. set of "critical values" of f, set of values $f(P)$ with $e_P(f) > 1$ (P a ramification point, a.k.a."critical point")

for each non-zero v in V. Observe that this is the same as a complex vector space structure on V extending the real one. So J is called a (linear) *complex structure* on V.

c) Verify that conversely, any complex line L has a canonical conformal structure and a canonical orientation, when viewed as a 2-dimensional real vector space.

d) Let V be a 2-dimensional real vector space, and, let $W = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$ the complexification of V. It is a 2-dimensional complex vector space, with a conjugation antilinear involution

$$w = v_1 + iv_2 \mapsto \overline{w} = v_1 - iv_2, \quad v_1, v_2 \in V.$$

Show that complex structures J on V are in natural bijection with complex lines $L \subset W$ such that $W = L \oplus \overline{L}$. Identify the set of these lines as the complement of $P(V) \simeq P^1(\mathbb{R})$ in $P(W) \simeq P^1(\mathbb{C})$, and show it is the union of two disjoint open hemispheres in $P(W)$. What distinguishes the two components ?

e) If $V = \mathbb{R}^2$, show that the set of conformal structures on V is in natural bijection with the (half-) hyperboloid

$$U = \left\{(a, b, c) \in \mathbb{R}^3 \mid ac - b^2 = 1, \mid a > 0\right\}.$$

In these coordinates, give an expression for the complex line L and complex structure J associated to $(a, b, c) \in U$ and the usual orientation of $\mathbb{R}^2$.

Exercise 5 Let $G = \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm 1\}$ be the group of automorphisms of $P^1(C)$, acting by Möbius transformations $z \mapsto (az + b)/(cz + d)$, $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$.

a) Show that $\mathrm{SL}_2(\mathbb{C})$ acts on the real vector space H of 2×2 hermitian matrices by $(g, h) \mapsto g^* hg$, preserving the quadratic form $\det : h \mapsto \det(h)$.

b) Show that det is of signature $(1, 3)$.

c) Deduce the existence of a group homomorphism

$$\rho : G = \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{SO}(1, 3).$$

d) Show that ρ is injective, and that its differential at the unit $e \in G$ is bijective.

e) Conclude that ρ defines an isomorphism

$$\rho : G = \mathrm{PSL}_2(\mathbb{C}) \to \rho(G) = \mathrm{SO}_0(1, 3),$$

the last group being the connected component of the identity in $\mathrm{SO}(1, 3)$.

f) Let $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$ be the subgroup of Möbius transformations $z \mapsto (az + b)/(cz+d)$, $a, b, c, d \in \mathbb{R}$ with $ad-bc = 1$. Show that it coincides with the subgroup preserving the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \subset P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

g) Verify that $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$ preserves the subspace $S \subset H$ (question **a)**) of 2×2 real symmetric matrices, and deduce that the group $\mathrm{PSL}_2(\mathbb{R})$ is isomorphic to $\mathrm{SO}_0(1, 2)$.

h) Define a bijection between the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and the set of (real) positive definite 2×2 matrices of determinant 1 which is compatible with the isomorphism of the previous question [3].

3. There is a relation with the last question in the previous exercise...