Exercise 1 Let D be a divisor on a compact connected Riemann surface X, and $L(D) = \{f \in \mathcal{M}(X) \mid f = 0 \text{ or } \operatorname{div}(f)+D \geq 0\}$ (it is a finite dimensional vector space of meromorphic functions on X).

a) Show that $L(D) = 0$ when $\deg(D) < 0$.

b) Show (or recall) that if D, D' are divisors on X with $D - D' = \operatorname{div}(g)$ for some g in $\mathcal{M}(X)^*$ (a principal divisor) then $L(D)$, $L(D')$ have the same dimension. In this situation, D, D' are said to be *linearly equivalent*, sometimes abbreviated $D \sim D'$. This (obviously) also implies $D_0 + D \sim D_0 + D'$ and $D_0 - D \sim D_0 - D'$ for any divisor D_0 on X.

c) Assuming $X = P^1(\mathbb{C})$, show that $\dim L(D) = 1 + \deg(D)$ if $\deg(D) \geq 0$ (one may reduce to the case $D = n[\infty]$).

Exercise 2 Let $E = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$.

a) Show that if P is a point of E, $\dim L([P]) = 1$ (and in fact $L([P]) = \mathbb{C}$, the constant functions ; one may use the residue theorem on E (first lesson)).

b) Recall that there is a meromorphic function (Weierstrass function) $\wp$ on E with a single pole at 0, which is of order 2, *i. e.* $\operatorname{ord}_0(\wp) = -2$. Show that if P is a point of E, $\dim L(2[P]) = 2$ (reduce to $P = 0$, and consider Laurent expansions). Generalize to $\dim L(n[P]) = n$, $n \geq 1$.

c) Show that if P, Q are points of E, $\dim L([P] + [Q]) = 2$ (reduce to the case $P = -Q \neq Q$, and recall that $\wp$ is even).

d) Recall Abel's theorem that a degree 0 divisor D on $\mathbb{C}/\Lambda$ is principal ($D \sim 0$) if and only if $A(D) = 0$, where A denotes the morphism of abelian groups $A : \operatorname{Div}(E) = \mathbb{Z}^{(E)} \to E = \mathbb{C}/\Lambda$ defined on basis elements by $[P] \mapsto P$. Show that if D is a divisor on E with $\deg D \geq 1$, the space $L(D)$ is of dimension $\deg D$ (reduce to the case $D = n[P]$).

e) If D is a divisor on E with $\deg(D) = 0$, the dimension of $L(D)$ is 0 except if $A(D) = 0 \in \mathbb{C}/\Lambda$ and then $\dim L(D) = 1$.

f) What is the degree of $\wp : E \to P^1(\mathbb{C})$? And of $\wp'$?

g) Give the value of $\operatorname{div}(\wp')$ (hint : $\wp'$ is an odd function).

Exercise 3 Let $(a, b; c, d) = (a-c)(b-d)/(a-d)(b-c)$ be (one of the possible definitions of) the cross-ratio[1] of four distinct complex numbers a, b, c, d.

a) Verify that on $P^1(\mathbb{C})$ the map $z \mapsto (a, b; c, z)$ is the homography sending a, b, c to $\infty, 0, 1$ respectively. Deduce that the cross-ratio of distinct complex numbers extends to distinct points of $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, and takes values in $\mathbb{C} \setminus \{0, 1\}$. Suggest an extension to not necessarily distinct points.

b) Show that by permuting the arguments of $(a, b; c, d)$ one obtains at most six values. If x is one of them, the others are $1/x, 1 - x, 1 - 1/x, 1/(1 - x), x/(x - 1)$ (hint : consider generators of the symmetric group on four letters).

c) Which group Γ of order 6 acting biholomorphically on $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ appears in the previous question ?

d) Show that $F(T) = (T^2 - T + 1)^3/(T(T - 1))^2 \in \mathbb{C}(T)$ verifies $F(1/T) = F(1 - T) = F(T)$ and deduce that $F((a, b; c; d))$ is invariant by permutations of a, b, c, d.

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e) Denote again by $F : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ the holomorphic map defined by F. What is its degree ? What are the points $P \in P^1(\mathbb{C})$ such that $e_P(F) > 1$? Show that the fibers $F^{-1}(y)$, $y \in P^1(\mathbb{C})$ are the orbits of the action found in question c).

f) Draw a picture of the group action found in question c) by showing that $D = \{z \in \mathbb{C} \mid |z| \leq 1 \text{ and } \mathrm{Re}(z) \geq 1/2\}$ is a fundamental domain, *i. e.* $P^1(\mathbb{C}) = \cup_{\gamma \in \Gamma} \gamma(D)$ and $\mathrm{int}(\gamma(D) \cap D) = \emptyset$ if $\gamma \in \Gamma \setminus \{1\}$ (hint : $\Gamma \subset \mathrm{Aut}(P^1(\mathbb{C}))$ sends circles to circles).