Exercise 1. Let $\varphi : X \to Y$ be a non-constant holomorphic map between compact connected Riemann surfaces X and Y.

a) If $h \in \mathcal{M}(Y)^\times$ is a non-zero meromorphic function on Y, $h \circ \varphi$ is a non-zero meromorphic function on X. Show that for $P \in X$,

$$\operatorname{ord}_P(h \circ \varphi) = e_P(\varphi) \operatorname{ord}_{\varphi(P)}(h).$$

b) Deduce that

$$\operatorname{div}(h \circ \varphi) = \sum_{Q \in Y} \operatorname{ord}_Q(h) \sum_{P \in \varphi^{-1}(Q)} e_P(\varphi)[P]$$

which can be written more concisely as $\operatorname{div}(h \circ \varphi) = \varphi^*(\operatorname{div}(h))$ for the morphism of abelian groups $\varphi^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ defined on free generators by $\varphi^*[Q] = \sum_{P \in \varphi^{-1}(Q)} e_P(\varphi)[P]$.

c) Similarly, if ω is a non-zero meromorphic form on Y, show that $\varphi^*(\omega)$ is a non-zero meromorphic form on X, and that

$$\operatorname{div}(\varphi^*(\omega)) = \varphi^*(\operatorname{div}(\omega)) + \sum_{P \in X} (e_P(\varphi) - 1)[P].$$

d) Recall that $\deg(\varphi) = \sum_{P \in \varphi^{-1}(Q)} e_P(\varphi)$ for any $Q \in Y$, and also that the degree of the divisor of any non-zero meromorphic form on a compact connected Riemann surface of genus g is $2g - 2$. Deduce the Riemann-Hurwitz formula

$$2g(X) - 2 = \deg(\varphi)\,(2g(Y) - 2) + \sum_{P \in X} (e_P(\varphi) - 1).$$

In particular, $g(X) \geq g(Y)$.

e) Let $F_n = \{(x, y) \in \mathbb{C}^2 \mid x^n + y^n = 1\}$. Show that F_n is a complex one dimensional submanifold of $\mathbb{C}^2$, hence a (non-compact) Riemann surface.

f) In the open subset $U = \{(x, y) \in \mathbb{C}^2 \mid |x| > 1\}$, consider the change of coordinates $x' = 1/x, y' = y/x$. More formally, show that $\psi : (x, y) \mapsto (1/x, y/x)$ defines a biholomorphism between U and $U' = \{(x', y') \in \mathbb{C}^2 \mid 0 < |x'| < 1\}$, and that $F_n \cap U$ is mapped by ψ onto the set $F'_n \cap U'$ where $F'_n = \{(x', y') \in \mathbb{C}^2 \mid 1 + y'^n = x'^n\}$ [1].

g) Conclude that there is a compact Riemann surface $\widehat{F}_n$ with $\widehat{F}_n \setminus F_n$ of cardinal n, and a holomorphic map $\varphi : \widehat{F}_n \to P^1(\mathbb{C})$ extending the first projection $(x, y) \mapsto x$ from F_n to $\mathbb{C}$ and mapping $\widehat{F}_n \setminus F_n$ to ∞.

h) Show that φ is ramified only over the n-th roots of unity $\mu_n = \{x \in \mathbb{C} \mid x^n = 1\}$, with $\varphi^{-1}(x) = \{(x, 0)\}$ and $e_{(x,0)}(\varphi) = n$ for each $x \in \mu_n$.

i) Denoting by $\mathbb{D}$ the open unit disk in $\mathbb{C}$, show that $\varphi^{-1}(\mathbb{D})$ and $\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$ have n connected components, and that $\mu_n \times \{0\}$ is contained in the closure of each of these components. Conclude that $\widehat{F}_n$ is connected.

j) Show that $g(\widehat{F}_n) = (n - 1)(n - 2)/2$. Verify directly that $\widehat{F}_2$ is isomorphic to $P^1(\mathbb{C})$.

k) Show that the formula $\omega = dx/y^{n-1}$ defines a meromorphic differential on $\widehat{F}_n$, which is in fact holomorphic on F_n and has as divisor $K = (n - 3) \sum_{P \in \widehat{F}_n \setminus F_n} [P]$.

1. one could also have taken $U = U' = \mathbb{C}^* \times \mathbb{C}$.

Exercise 2. Let $P \in \mathbb{C}[X]$ be a polynomial of degree n with distinct roots, and $X = \{(x, y) \in \mathbb{C}^2 \mid y^m = P(x)\}$.

a) Show that X is a one dimensional complex submanifold of $\mathbb{C}^2$, hence a (non-compact) Riemann surface.

b) For a large enough $R > 0$ construct a biholomorphism from $X_R = X \cap \{(x, y) \in \mathbb{C}^2 \mid |x| > R\}$ to the curve $X' = \{(x', y') \in (U' \setminus \{0\}) \times \mathbb{C} \mid y'^m = x'^n\} \subset \mathbb{C}^* \times \mathbb{C}^*$, where U' is a neigbourhood of 0 in $\mathbb{C}$.

c) Show that if a, b, p, q are integers, and $aq - bp = 1$, the map $(u, v) \mapsto (u^a v^b, u^p v^q)$ is a biholomorphism of $\mathbb{C}^* \times \mathbb{C}^*$ (and a group automorphism), with inverse of the same form.

d) Returning to question **??**, assume that m, n are relatively prime. Show that the map $\gamma : t \mapsto (t^m, t^n)$ is a biholomorphism from a pointed neighbourhood of 0 in $\mathbb{C}$ to X', and use this to construct a compact connected Riemann surface $\widehat{X} = X \sqcup \{P_\infty\}$ with a holomorphic map φ of degree m to $P^1(\mathbb{C})$ such that $\varphi^{-1}(\infty) = \{P_\infty\}$.

e) Using Riemann-Hurwitz formula show that the genus of $\widehat{X}$ is $(m - 1)(n - 1)/2$.

f) No longer assuming m, n relatively prime, let $m = m'\delta, n = n'\delta$, where $\delta = \gcd(m, n)$. Show that X' is biholomorphic to a disjoint union of δ pointed neighborhoods of 0 in $\mathbb{C}$.

g) Use this to construct a compact connected Riemann surface $\widehat{X} = X \sqcup S_\infty$ with $\mathrm{card}(S_\infty) = \delta$, together with a holomorphic map φ of degree m to $P^1(\mathbb{C})$ such that $\varphi^{-1}(\infty) = S_\infty$.

h) Use Riemann-Hurwitz formula to show that the genus of $\widehat{X}$ is $((m-1)(n-1)-(\delta-1))/2$.

i) Check that the formula dx/y^{m-1} defines a meromorphic 1-form ω on $\widehat{X}$, with no poles or zeros on X. If m, n are relatively prime (so that $S_\infty = \{P_\infty\}$), verify that $\mathrm{ord}_{P_\infty}(\omega) = mn - m - n - 1$. Discuss the case $m = 2, n = 3$.

Exercise 3. Let V be a complex vector space of finite dimension n, and consider for $k \in \mathbb{N}$ the complex vector space $A^k_{\mathbb{R}}(V)$ of $\mathbb{R}$-multilinear antisymmetric maps $\alpha : V^k \to \mathbb{C}$, $(v_1, \dots, v_k) \mapsto \alpha(v_1, \dots, v_k)$.

a) For $k = 1$ show that $A^1_{\mathbb{R}}(V) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is the direct sum its subspaces V^* and $\overline{V}^*$ of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear maps $V \to \mathbb{C}$. Here $\overline{V}$ denotes V with the conjugate complex structure $v \mapsto -iv$, with elements written $\overline{v}$ to avoid confusion, and an antilinear "identity map" $v \mapsto \overline{v}$ from V to $\overline{V}$.

b) Show that $A^k_{\mathbb{R}}(V)$ is isomorphic to the space $A^k_{\mathbb{C}}(V \oplus \overline{V})$ of $\mathbb{C}$-multilinear antisymmetric maps $(V \oplus \overline{V})^k \to \mathbb{C}$. More precisely, show that composition with the product map $\Delta^k : V^k \mapsto (V \oplus \overline{V})^k$, $\Delta(v) = (v \oplus \overline{v})/2$ induces an isomorphism $A^k_{\mathbb{C}}(V \oplus \overline{V}) \to A^k_{\mathbb{R}}(V)$ (remark that $\Delta(e_1), \Delta(ie_1), \dots, \Delta(e_n), \Delta(ie_n)$ constitute a basis of the complex vector space $V \oplus \overline{V}$). We will write $\tilde{\alpha} \in A^k_{\mathbb{C}}(V \oplus \overline{V})$ the map sent to $\alpha \in A^k_{\mathbb{R}}(V)$ by this isomorphism.

c) For $p, q \in \mathbb{N}$, $p + q = k$, and $\tilde{\alpha} \in A^k_{\mathbb{C}}(V \oplus \overline{V})$, denote by $\tilde{\alpha}_{p,q}$ the restriction of $\tilde{\alpha}$ to $V^p \times \overline{V}^q$. This is a $\mathbb{C}$-multilinear map which is antisymmetric in the first p and last q variables. Show that the maps $\tilde{\alpha}_{p,q}$ for $p, q \in \mathbb{N}$, $p + q = k$ determine $\tilde{\alpha}$.

d) Conversely, show that a $\mathbb{C}$-multilinear map $\beta : V^p \times \overline{V}^q \to \mathbb{C}$ which is antisymmetric in the first p and last q variables determines a (unique) map $\tilde{\alpha} \in A^k_{\mathbb{C}}(V \oplus \overline{V})$ whith $\tilde{\alpha}_{p,q} = \beta$ and $\tilde{\alpha}_{p',q'} = 0$ for $(p', q') \neq (p, q)$.

e) Show that the vector subspace $A^{p,q}(V) \subset A^k_{\mathbb{R}}(V)$ of maps α such that $\tilde{\alpha}_{p',q'} = 0$ for $(p', q') \neq (p, q)$ is characterized by the condition $\alpha(\lambda v_1, \dots, \lambda v_k) = \lambda^p \overline{\lambda}^q \alpha(v_1, \dots, v_k)$ for all $\lambda \in \mathbb{C}$ (or $\mathbb{S}^1$) and all $v_1, \dots, v_k$ in V.

f) Show that $A^k_{\mathbb{R}}(V) = \bigoplus_{p+q=k} A^{p,q}(V)$.
g) Show that $A^{p,q}(V) \neq 0$ only if $0 \leq p, q \leq k$ and $p + q = k$.