Exercise 1.

a) Let $g : U \to U$ be a C^1 diffeomorphism of a neighbourhood U of the origin $0 \in \mathbb{R}^d$. Assume that $g(0) = 0$ and g is of finite order dividing n (meaning $g^n = \mathrm{id}_U$). Letting $A = Dg(0) \in GL_d(\mathbb{R})$, show that $A^n = \mathrm{Id}$ and that the formula

$$\Phi(x) = \frac{1}{n} \sum_{0 \le k < n} A^{-k} g^k(x)$$

defines a map $\Phi; U \to \mathbb{R}^d$ such that $\Phi \circ g = A \circ \Phi$, and that Φ restricts to a C^1-diffeomorphism on a neigbourhood of 0 ("g is C^1-linearizable near 0").

b) Show that for any integer m dividing n, the set U_m of points $x \in U$ such that $g^m = \mathrm{id}$ in a neighbourhood of x is open and closed in U (apply the previous question to g^m near a point of $\overline{U_m} \setminus U_m$).

c) Deduce that if U is connected and $g^k \neq \mathrm{id}_U$ for $0 < k < n$, A is also of exact order n (hint : any neighbourhood of 0 contains points of exact period n under g).

d) In the particular case where $d = 2$, $\mathbb{R}^2 \simeq \mathbb{C}$ and g is holomorphic, verify that Φ is also holomorphic near 0 and conjugates g to the map $w \mapsto \lambda w$ for some n-root of unity $\lambda \in \mathbb{C}$. If n is the smallest integer with this property, λ is a primitive n-root of unity. In particular the g-invariant holomorphic functions f defined near 0 are then of the form $f = \tilde{f} \circ \pi_n$, where $\pi_n(z) = z^n$ and $\tilde{f}$ is holomorphic near 0.

Exercise 2. Recall that $\mathrm{Aut}(P^1(\mathbb{C})) \simeq \mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$ is the group of homographic transformations $z \mapsto (az+b)/(cz+d)$ of $\mathbb{C} \cup \{\infty\}$, with $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. One may reduce to $ad - bc = 1$, that is $\mathrm{PGL}_2(\mathbb{C}) \simeq \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm 1\}$.

a) If $G \subset \mathrm{PSL}_2(\mathbb{C})$ is a discrete subgroup acting properly on $P^1(\mathbb{C})$, show that G is finite.

b) For G a finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$, let $\tilde{G}$ denote its inverse image in $\mathrm{SL}_2(\mathbb{C})$. It is also finite. Show that there is a $P \in \mathrm{GL}_2(\mathbb{C})$ such that $P\tilde{G}P^{-1} \subset \mathrm{SU}(2)$ (hint : take any hermitian metric on $\mathbb{C}^2$ and average it under G).

c) Show that any element of order n in $\mathrm{SU}(2)$ is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, with λ a primitive n-th root of 1 in $\mathbb{C}$. This matrix acts on $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ as $z \mapsto \lambda^2 z$. In particular $-\mathrm{id}$ is the only element of order 2. It acts trivially on $P^1(\mathbb{C})$.

d) Show that the action of $\mathrm{SU}(2)/\{\pm 1\} \subset \mathrm{PSL}_2(\mathbb{C})$ on $P^1(\mathbb{C})$ is sent by stereographic projection $z \mapsto (2z/(|z|^2+1), (|z|^2-1)/(|z|^2+1)) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$ to the usual action of $\mathrm{SO}(3)$ on the unit sphere $\mathbb{S}^2$ (hint : show that the induced action on $\mathbb{S}^2$ is by linear transformations of $\mathbb{R}^3$). Hence any finite subgroup of $\mathrm{Aut}(P^1(\mathbb{C}))$ is isomorphic to a finite subgroup of $\mathrm{SO}(3)$ [1].

e) Let G be a finite subgroup of $\mathrm{Aut}(P^1(\mathbb{C}))$, $X = P^1(\mathbb{C})$, $Y = X/G$ the quotient Riemann surface and $\pi : X \to Y$ the quotient map. The Riemann-Hurwitz formula reads

$$-2 = (2g(Y) - 2)\deg(\pi) + \sum_{P \in X} (e_P(\pi) - 1)$$

with $\deg(\pi) = |G|$ (the order of G) and $e_P(\pi) = |G_P|$ (the order of the stabilizer of P). Necessarily $g(Y) = 0$ [2]. Show that all points P in the same orbit/fiber $\pi^{-1}(Q)$ have the same

1. A simple proof results from consideration of the action of SU(2) on hermitian 2 by 2 matrices of trace 0.
2. this implies that Y is isomorphic to $P^1(\mathbb{C})$, either by Riemann-Roch for the holomorphic definition of genus or by uniformization for the topological one.

ramification index m_Q, and that their number is $|G|/m_Q$.

f) Let $S \subset Y$ be the finite set of points Q with $m_Q > 1$. Show that S has cardinal $0, 2$ or 3, with the first two cases corresponding to G trivial or cyclic. In case $|S| = 3$, show that the possibilities for the three integers m_1, m_2, m_3 are, up to reordering, $2, 2, n$ ($n \geq 2$), $2, 3, 3$, $2, 3, 4$ and $2, 3, 5$. To which groups G can you relate these possibilities ? [3]

Exercise 3. Recall that $\mathrm{SL}_2(\mathbb{R})$ acts (non-faithfully) on the upper half-plane $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by $\gamma \cdot z = (az+b)/(cz+d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Let $G \subset \mathrm{SL}_2(\mathbb{R})$ be a subgroup which is discrete for the (usual) topology induced by the embedding $\mathrm{SL}_2(\mathbb{R}) \subset M_2(\mathbb{R}) \simeq \mathbb{R}^4$.

a) Show that G is closed in $\mathrm{SL}_2(\mathbb{R})$, and also in $M_2(\mathbb{R})$.

b) Show that the action of G on the upper half-plane H is proper, first by admitting that the action of $\mathrm{SL}_2(\mathbb{R})$ on H is proper.

c) Show that the action of $\mathrm{SL}_2(\mathbb{R})$ is proper (hint : first show that the stabilizer of $i \in H$ is compact. Then find a sequence of compact subsets K_n of $\mathrm{SL}_2(\mathbb{R})$ such that the subsets $K_n \cdot i$ exhaust H) [4].

d) Show that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. What are the possible orders of finite order elements of Γ ?

e) Show that the set $F = \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \leq 1/2, |z| \geq 1\}$ meets each orbit of Γ on H (hint : for fixed $z \in H$, maximize $\mathrm{Im}(\gamma(z))$, $\gamma \in \Gamma$, and use the elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Γ) [5]

f) Show that if a holomorphic function $f : H \to \mathbb{C}$ is Γ-invariant it is necessarily of the form

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q^n = \sum_{n \in \mathbb{Z}} a_n \exp(2i\pi n z)$$

where $q = \exp(2i\pi z)$ lies in the pointed unit disk $\mathbb{D}^*$ and moreover $f(-1/z) = f(z)$ for all $z \in H$.

3. see pages 80-82 in R. Miranda - Algebraic curves and Riemann surfaces - AMS 1995
4. The real reason for properness is that the action is by isometries for a complete riemannian metric on H – the Poincaré metric.
5. search "keith conrad sl2z" on the web if you are stuck.