Exercise 1. (recycled) Recall that $\mathrm{SL}_2(\mathbb{R})$ acts (non-faithfully) on the upper half-plane $H = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ by $\gamma \cdot \tau = \gamma(\tau) = (a\tau + b)/(c\tau + d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Let $G \subset \mathrm{SL}_2(\mathbb{R})$ be a subgroup which is discrete for the (usual) topology induced by the embedding $\mathrm{SL}_2(\mathbb{R}) \subset M_2(\mathbb{R}) \simeq \mathbb{R}^4$.

a) Show that G is closed in $\mathrm{SL}_2(\mathbb{R})$, and also in $M_2(\mathbb{R})$.

b) Show that the action of G on the upper half-plane H is proper, first by admitting that the action of $\mathrm{SL}_2(\mathbb{R})$ on H is proper.

c) Show that the action of $\mathrm{SL}_2(\mathbb{R})$ is proper (hint : first show that the stabilizer of $i \in H$ is compact. Then find a sequence of compact subsets K_n of $\mathrm{SL}_2(\mathbb{R})$ such that the subsets $K_n \cdot i$ exhaust H) [1].

d) Show that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. What are the possible orders of finite order elements of Γ ?

e) Show that the set $F = \{\tau \in \mathbb{C} \mid |\mathrm{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$ meets each orbit of Γ on H (hint : for fixed $\tau \in H$, maximize $\mathrm{Im}(\gamma(\tau))$, $\gamma \in \Gamma$, and use the elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Γ)

f) Show that in the preceding question, one proved in fact that F meets each orbit of the subgroup $\tilde{\Gamma} \subset \Gamma$ generated by S and T. By considering the Γ-orbit of the point $2i \in H$, deduce that $\Gamma = \tilde{\Gamma}$, that is $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T (this can also be deduced from the termination of the euclidean algorithm on $\mathbb{Z}$).

g) Show that if a holomorphic function $f : H \to \mathbb{C}$ is T-invariant it is necessarily of the form

$$f(\tau) = \varphi(q) = \sum_{n \in \mathbb{Z}} a_n q^n = \sum_{n \in \mathbb{Z}} a_n \exp(2i\pi n\tau)$$

where $q = \exp(2i\pi\tau)$ lies in the pointed unit disk $\mathbb{D}^*$ and φ is holomorphic there. What must the $|a_n|$ verify for $n \to \pm\infty$? The function f is then Γ-invariant if and only if one has moreover $f(-1/\tau) = f(\tau)$ for all $\tau \in H$. It is then called a modular function [2].

h) One can show that as a Riemann surface, H/Γ is isomorphic to $\mathbb{C}$, with coordinate given by the modular invariant $j(\tau) = 1/q + 744 + 196884\,q + \ldots$, a q-series with integral coefficients. A holomorphic function $f : H \to \mathbb{C}$ is called a (weak) modular form of weight $2k$ under Γ if the expression $f(\tau)\,(d\tau)^k$ is formally invariant by Γ, which writes

$$f(\gamma(\tau))\,\gamma'(\tau)^k = f(\tau),\ \tau \in H,\ \gamma \in \Gamma$$

or more explicitly

$$f((a\tau + b)/(c\tau + d)) = (c\tau + d)^{2k} f(\tau),\ a, b, c, d \in \mathbb{Z}, ad - bc = 1,\ \tau \in H.$$

Show that this is equivalent to the conjunction of $f(\tau+1) = f(\tau)$ and $f(-1/\tau) = \tau^{2k} f(\tau)$ for $\tau \in H$. In the case $k = 1$, (weak) modular forms of weight 2 are identified to holomorphic 1-forms on H/Γ. For more on this subject, one can consult Serre's "Cours d'arithmétique" or Diamond

1. The real reason for properness is that the action is by isometries for a complete riemannian metric on H, the Poincaré metric $|d\tau|^2/\mathrm{Im}(\tau)^2$.
2. This comes from Γ being called the "modular group"

and Shurman's "A first course in modular forms" (2005), or James Milne "Modular Functions and Modular Forms" (available at `http://www.jmilne.org/math/CourseNotes/mf.html`).

i) (examples) Let

$$f_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} (m\tau+n)^{-2k}\ ,\ \tau\in H.$$

Show that f_k is a modular form of weight $2k$ under Γ. We saw in the construction of the Weierstrass function $\wp(z;\tau)$ (with period lattice $\mathbb{Z}+\tau\mathbb{Z}$) that $g_2(\tau)=60f_2(\tau)$ and $g_3(\tau)=140f_3(\tau)$ were coefficients for the differential equation $(d\wp/dz)^2=4\wp^3-g_2\wp-g_3$ satisfied by $\wp$. The discriminant $\Delta=g_2^3-27g_3^2$ is then a weight 12 modular form, with no zeros on H, and one can show that the modular invariant $j(\tau)$ alluded to above is $1728g_2^3/\Delta$. It is a ("the") modular function under Γ.

Exercise 2. (Another source of modular forms) Let $f:\mathbb{R}\to\mathbb{C}$ be a function which is smooth and rapidly decreasing, along with all its derivatives. Define the Fourier transform of f as

$$\hat{f}(\xi)=\int_{\mathbb{R}} e^{-2i\pi\xi x}f(x)\,dx.$$

Recall that a smooth function $F:\mathbb{R}/\mathbb{Z}\to\mathbb{C}$ is the sum of its Fourier series $F(x)=\sum_{n\in\mathbb{Z}}a_ne^{2i\pi nx}$, where

$$a_n=\int_{\mathbb{R}/\mathbb{Z}} e^{-2i\pi nx}F(x)\,dx.$$

a) Show that by taking $F(x)=\sum_{m\in\mathbb{Z}}f(x+m)$ and expressing that $F(0)=\sum_{n\in\mathbb{Z}}a_n$, one obtains the *Poisson summation formula*

$$\sum_{n\in\mathbb{Z}}\hat{f}(n)=\sum_{m\in\mathbb{Z}}f(m).$$

b) For λ real and non-zero, show that

$$\sum_{n\in\mathbb{Z}}\hat{f}(n\lambda)=\frac{1}{\lambda}\sum_{m\in\mathbb{Z}}f(m/\lambda).$$

c) Show that for all $\xi\in\mathbb{R}$, $\int_{\mathbb{R}}\exp(2i\pi\xi x-\pi x^2)dx=\exp(-\pi\xi^2)$ by considering a contour integral in the complex domain after having written the integrand as $\exp(-\pi(x+i\xi)^2)\exp(-\pi\xi^2)$.

d) Using that $f(x)=e^{-\pi x^2}$ is its own Fourier transform (preceding question), deduce that if $a\in\mathbb{C}$ and $\mathrm{Re}(a)>0$,

$$\sum_{n\in\mathbb{Z}}e^{-\pi n^2a}=\frac{1}{\sqrt{a}}\sum_{m\in\mathbb{Z}}e^{-\pi m^2/a}$$

where $\sqrt{\cdot}$ denotes the determination taking $\mathbb{R}_+$ into itself.

e) For $\mathrm{Im}(z)>0$, define $q=\exp(i\pi z)\in\mathbb{D}^*$ and $\theta(z)=\sum_{n\in\mathbb{Z}}q^{n^2}$. Show that $\theta(-1/z)=\sqrt{z/i}\,\theta(z)$ (and "only" $\theta(z+2)=\theta(z)$).