Exercise 1. Let $\mathcal{R}$ be the set of lattices in $\mathbb{R}^2$, that is the set of (closed) discrete subgroups $R \subset \mathbb{C}$ isomorphic to $\mathbb{Z}^2$ (or equivalently subgroups generated by a vector space basis of $\mathbb{R}^2$ over $\mathbb{R}$).

a) Show that the map $\mathrm{GL}_2(\mathbb{R}) \to \mathcal{R}$ defined by $A \mapsto A(\mathbb{Z}^2)$ identifies $\mathcal{R}$ to the quotient set $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2(\mathbb{Z})$. We put on $\mathcal{R}$ the corresponding quotient topology.

b) Show that the action $\mathrm{GL}_2(\mathbb{R}) \times \mathcal{R} \to \mathcal{R}$ is continuous.

c) Show that $\mathcal{R}$ has a unique manifold structure (of dimension 4) such that the projection $\pi : \mathrm{GL}_2(\mathbb{R}) \to \mathcal{R}$ is a local diffeomorphism.

d) Identify now $\mathbb{R}^2$ with $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, and consider the action of $\mathbb{C}^*$ on $\mathcal{R}$ by multiplication $(c, R) \mapsto cR$, with quotient denoted $\mathbb{C}^* \backslash \mathcal{R}$, and quotient map $R \mapsto [R]$. Show that this action is proper. What are its stabilizers ?

e) Show that the map $q : H = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \to \mathbb{C}^* \backslash \mathcal{R}$, defined by $q(\tau) = [\mathbb{Z} \oplus \tau\mathbb{Z}]$ is surjective, and identifies $\mathbb{C}^* \backslash \mathcal{R}$ with the quotient of H by the proper action of a discrete group $\Gamma \simeq \mathrm{PSL}_2(\mathbb{Z})$ of holomorphic automorphisms $\tau \mapsto (a\tau + b)/(c\tau + d)$, $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ (hint : write the condition for $(\tau', 1)$ and $(\lambda\tau, \lambda)$, $\lambda \in \mathbb{C}^*$, to be bases of the same lattice).

f) For $k \in \mathbb{Z}$, define

$$F_k = \{f : \mathcal{R} \to \mathbb{C} \mid f(cR) = c^{-k} f(R), c \in \mathbb{C}^*, R \in \mathcal{R}\}.$$

Show that F_k is reduced to 0 if k is odd. Show that if k is even F_k can be identified with the space Φ_k of functions $\varphi : H \to \mathbb{C}$ verifying $\varphi(\tau) = \varphi(\gamma(\tau))\, \gamma'(\tau)^{k/2}$ for all $\gamma \in \Gamma$, $\tau \in H$ (hint : for $f \in F_k$, consider $\varphi(\tau) = f(\mathbb{Z} + \tau\mathbb{Z})$).

In particular smooth functions in F_0 (resp. F_2) are identified with smooth Γ-invariant functions (resp. $(1, 0)$-forms) on H.

g) Show that for (even) $k > 2$, the formula $G_k(R) = \sum_{\lambda \in R \setminus \{0\}} 1/\lambda^k$ defines a element of F_k. We already encountered G_4, G_6 in the study of Weierstrass elliptic functions, as functions of $\tau \in H$. We will still denote (abusively) by $G_k \in \Phi_k$ the holomorphic functions on H defined by

$$G_k(\tau) = G_k(\mathbb{Z} + \tau\mathbb{Z}) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m\tau + n)^k}.$$

Being holomorphic on H and invariant by $\tau \mapsto \tau + 1$, these functions possess Laurent developments in the variable $q = \exp(2i\pi\tau)$, converging for $0 < |q| < 1$. In the following we will compute (some of) these developments.

h) Show that the formula

$$E(z) = 1/z + \sum_{n \geq 1} \left(\frac{1}{z + n} + \frac{1}{z - n}\right)$$

defines a meromorphic function E on $\mathbb{C}$ which is $\mathbb{Z}$-periodic and odd, and moreover that $E(z)$ tends to non-zero (opposite) limits when $\mathrm{Im}(z) \to \pm\infty$.

i) Deduce that $E(z) = \varphi(q)$ for $q = \exp(2i\pi z)$ where φ is a meromorphic function on $\mathbb{C}^*$ with only a simple pole at $q = 1$, a simple zero at $q = -1$, and which is bounded near 0 and ∞. Conclude that $\varphi(q) = i\pi\frac{q+1}{q-1}$ (e.g. by computing the residue of $\varphi(q)dq$ at $q = 1$) so that

$$E(z) = \pi \cos(\pi z)/\sin(\pi z)$$

for $z \in \mathbb{C} \setminus \mathbb{Z}$ and

$$\varphi(q) = -i\pi(1 + 2\sum_{l \geq 1} q^l)\ , \ q \in \mathbb{C}, |q| < 1.$$

j) For $k \geq 2$, and $\mathrm{Im}(z) > 0$, $q = \exp(2i\pi z)$, deduce that

$$\sum_{n\in\mathbb{Z}} 1/(z+n)^k = \frac{(-2i\pi)^k}{(k-1)!} \sum_{l\geq 1} l^{k-1} q^l$$

(hint : $d/dz = 2i\pi\, q\, d/dq$), and then that for even integers $k > 2$

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(m\tau+n)^k} = 2\sum_{n\geq 1} 1/n^k + 2\frac{(-2i\pi)^k}{(k-1)!} \sum_{m\geq 1}\sum_{l\geq 1} l^{k-1} q^{ml}$$

so that denoting $\zeta(k) = \sum_{n\geq 1} 1/n^k$, $\sigma_{k-1}(m) = \sum_{l|m} l^{k-1}$, one has for any even $k > 2$

$$G_k(\tau) = 2\zeta(k) + 2\frac{(-2i\pi)^k}{(k-1)!} \sum_{m\geq 1} \sigma_{k-1}(m) q^m.$$

k) Returning to the function E of question h), define rational (Bernoulli) numbers B_k by

$$\beta(t) = t/(e^t - 1) = \sum_{k\geq 0} B_k\, t^k/k! = (1 + t/2 + t^2/6 + \ldots)^{-1} = 1 - t/2 + t^2/12 + \ldots).$$

Show that

$$E(z) = i\pi + \frac{1}{z}\beta(2i\pi z),$$

and deduce that $\zeta(k)$ for even $k \geq 2$ is a rational multiple of π^k

$$2\zeta(k) = -\frac{(2i\pi)^k}{k!} B_k.$$

Then conclude that for even $k \geq 4$,

$$\frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{m\geq 1} \sigma_{k-1}(m) q^m$$

l) Recall from the study of Weierstrass elliptic functions that the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, $\tau \in H$, is isomorphic, when punctured at 0, to the plane cubic curve

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$$

in $\mathbb{C}^2$ with $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 140G_6(\tau)$.

Show that the discriminant of the polynomial $P_\tau(x) = 4x^3 - g_2(\tau)x - g_3(\tau)$ is a constant multiple of $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, and that the holomorphic function Δ on H belongs to the space Φ_{12}.

m) Check that $\Delta(\tau)$ has a q-development without constant term

$$\Delta(\tau) = \sum_{m\geq 1} a_m q^m$$

with moreover $a_1 = (2\pi)^{12}$ (hint : $B_4 = -1/30$, $B_6 = 1/42$).

n) Using the properties of Weiertrass function $\wp_\tau$, show that Δ has no zero on H.

o) Conclude that the function $\tau \mapsto j(\tau) = g_2^3(\tau)/\Delta(\tau)$ is holomorphic on H and invariant by Γ. One can show with some more work that it defines a biholomorphism from the quotient Riemann surface H/Γ to $\mathbb{C}$, known as the modular j-invariant [1].

1. See for instance Serre's "A course in arithmetic", or Diamond and Shurman's "Introduction to modular forms", or James Milne "Modular Functions and Modular Forms" on his webpage. The usual normalization of j is $12^3 = 1728$ times this.