**Exercise 1**. Let $f(X) \in \mathbb{C}[X]$ be a polynomial of degree $n$ such that the zeros $a_1, \ldots, a_{n-1}$ of its derivative $f'(X)$ are simple and the corresponding critical values $b_1 = f(a_1), \ldots, b_{n-1} = f(a_{n-1})$ are distinct (a "Morse polynomial").

**a)** Show that the holomorphic map $f : \mathbb{C} \to \mathbb{C}$ is of degree $n$ with branching set $B = \{b_1, \ldots, b_{n-1}\}$. What are the cardinals of the fibers $f^{-1}(b)$, $b \in B$ ?

**b)** Consider $Y = \mathbb{C} \setminus B$ and the restriction $p = f|_Y : X = f^{-1}(Y) \to Y$. It is a covering map of degree $n$ (by the inverse function theorem), and choosing a base point $y_0 \in Y$ one defines the monodromy (permutation) representation

$$\rho : \pi_1(Y, y_0) \to \mathrm{Sym}(p^{-1}(y_0)) \simeq S_n.$$

More precisely, the unique lifting property of continuous paths in $Y$ defines for any $\gamma : [0,1] \to Y$ a "monodromy" bijection $\rho(\gamma) : f^{-1}(\gamma(0)) \to f^{-1}(\gamma(1))$ which only depends on the homotopy class of $\gamma$ (with fixed ends).

The concatenation of (homotopy classes) of paths is sent by $\rho$ to the composition of monodromies.

For any two base points $y_0$, $y_1$ in $Y$ (in the same path component), the choice of a (homotopy class of) path $\lambda$ from $y_0$ to $y_1$ gives rise to an isomorphism $\pi_1(Y, y_1) \to \pi_1(Y, y_0)$ obtained by pre- and post- concatenating any loop $\gamma$ based at $y_1$ with $\lambda$ and its reverse $\overline{\lambda}$. Symbolically $\gamma \mapsto \overline{\lambda}\gamma\lambda$ [1]. If the path $\lambda$ is replaced by $\mu$, say, the homotopy class of $\overline{\lambda}\gamma\lambda$ in $\pi_1(Y, y_0)$ changes by *conjugation* by the homotopy class of $\overline{\mu}\lambda$ in $\pi_1(Y, y_0)$,

$$\overline{\mu}\gamma\mu \sim \overline{\mu}\lambda\overline{\lambda}\gamma\lambda\overline{\lambda}\mu = (\overline{\mu}\lambda)(\overline{\lambda}\gamma\lambda)\overline{(\overline{\mu}\lambda)} .$$

Without choosing a path from $y_0$ to $y_1$, we see that this associates to $\gamma$ based at $y_1$ a conjugacy class in $\pi_1(Y, y_0)$.

Show that if $s_i \in \pi_1(Y, y_0)$ is in the conjugacy class defined by a small circle around $b_i = f(a_i)$ $\rho(s_i)$ is a transposition in $S_n$, for each $i = 1, \ldots, n-1$.

**c)** Denote by $G \subset S_n$ the image of $\rho$. Deduce from the connectedness of $X = \mathbb{C} \setminus f^{-1}(B)$ that $G$ is transitive on $\{1, \ldots, n\}$.

**d)** Show that if $s \in \pi_1(Y, y_0)$ is in the conjugacy class defined by a large circle encircling all $f(a_i)$, $\rho(s)$ is a cycle of length $n$ in $S_n$ (thus re-proving the transitivity of $G$).

**e)** Conclude that $G = S_n$ (hint : show by suitable choice of $y_0$ and representative loops of $s_1, \ldots, s_{n-1}, s$ that the product of $s_1, \ldots, s_{n-1}$ in some order is equal to $s$ ; deduce that the transpositions $\rho(s_i)$ generate $S_n$ from the lemma : if $n-1$ transpositions in $S_n$ have a cycle of length $n$ as product, they generate $S_n$) [2].

**f)** Show that the polynomial $f(X) = X^n - X$ verifies the hypotheses (is a "Morse polynomial").

**g)** Show that in the space $\mathbb{C}^n$ of monic degree $n$ polynomials $f = X^n + c_1X^{n-1} + \cdots + c_n$, the subset of Morse polynomials is dense and open (hint : find a polynomial in the coefficients of $f$ which vanishes if and only if $f$ is not Morse).

**Exercise 2**. Consider the polynomial $P(T, X) = X^n - nTX + 1 \in \mathbb{C}[T, X]$.

1. Here concatenation is denoted in the "compositional" order, which makes $\rho$ a morphism $\rho(\gamma_2\gamma_1) = \rho(\gamma_2)\rho(\gamma_1)$ rather than an anti-morphism.

2. In fact a better result holds : if $G \subset S_n$ is transitive and generated by transpositions, $G = S_n$, see J.-P. Serre "Topics in Galois theory", p. 40.

**a)** Show that $V = \{(t,x) \in \mathbb{C}^2 \mid P(t,x) = 0\}$ is a 1-dimensional complex submanifold of $\mathbb{C}^2$, hence a (non-compact) Riemann surface.

**b)** Show that $V$ is connected, and that the function $f : V \to \mathbb{C}$, $f(t,x) = t$ is a ramified covering of degree $n$. Determine the set of its branching points $B \subset \mathbb{C}$

**c)** Show that around each branching point $b \in B$ of $f$, the monodromy is a transposition.

**d)** Conclude that the monodromy $\rho : \pi_1(\mathbb{C} \setminus B, *) \to S_n$ is surjective (as in the previous exercise).

**Exercise 3**. Let $f(X,Y) = Y^n - P(X) \in \mathbb{C}[X,Y]$, where $P$ is a polynomial of degree $m \geq 1$ with simple roots.

**a)** Show that $V = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$ is a Riemann surface, that the restriction of the first projection $p_1 : V \to \mathbb{C}$ a ramified covering of degree $n$, and determine its branching set $B \subset \mathbb{C}$.

**b)** Show that the monodromy $\rho : \pi_1(\mathbb{C} \setminus B, *) \to S_n$ has for image a cyclic group of order $n$ (hint : consider the simply connected set $\Omega = \mathbb{C} \setminus (B + e^{i\alpha}\mathbb{R}_+)$ for a "generic" angle $\alpha$ and how the $n$ determinations of $(P(x))^{1/n}$ on $\Omega$ behave along the cuts $b + e^{i\alpha}\mathbb{R}_+$, $b \in B$).

**Exercise 4**. Let $B = \{b_1, \ldots, b_r\} \subset P^1(\mathbb{C})$ be a finite subset, $n \geq 2$ an integer, and for each $i = 1, \ldots, r$ let $\sigma_i \in S_n$ be a permutation.

Assume chosen a basepoint $o \in P^1(\mathbb{C}) \setminus B$, and segments $\lambda_i \simeq [0,1]$, $i = 1, \ldots, r$ from $o$ to $b_i$, pairwise disjoint outside $o$ and ordered by $1, 2, \ldots, r$ around $o$.

Denote by $s_i \in \pi_1(P^1(\mathbb{C}) \setminus B, o)$ the homotopy class of a loop first following $\lambda_i$, going once around $b_i$, and returning to $o$ along $\overline{\lambda_i}$.

**a)** Show that if there exists a ramified covering $f : X \to P^1(\mathbb{C})$ with branching set $B$ and monodromy $\rho : \pi_1(P^1(\mathbb{C}) \setminus B, o) \to S_n$ satisfying $\rho(s_i) = \sigma_i$, $i = 1, \ldots, r$, the product $\sigma_r \ldots \sigma_1$ is the identity in $S_n$.

Conversely, this condition is sufficient for the existence of (non-compact) surface $X_0$ and a covering map $f_0 : X_0 \to Y_0 = P^1(\mathbb{C}) \setminus B$ with monodromy $\rho$, which amounts to say that the fundamental group of $P^1(\mathbb{C}) \setminus \{b_1, \ldots, b_r\}$ is generated by $s_1, \ldots, s_r$ with the "only relation" $s_r \ldots s_1 = 1$ [3].

The surface $X_0$ has a unique Riemann surface structure making the covering map $f_0$ holomorphic. The covering $f_0 : X_0 \to Y_0 = P^1(\mathbb{C}) \setminus B$ restricted to the preimage of a small pointed disk $D_i^*$ around $b_i \in B$ is determined by the decomposition of the permutation $\sigma_i = c_{i,1} \ldots c_{i,m_i}$ as a product of disjoint cycles. Namely, $f_0^{-1}(D_i^*)$ is the disjoint union of punctured disks $D_{i,1}^*, \ldots D_{i,m_i}^*$, with $f_0|_{D_{i,j}^*}$ modeled on $z \mapsto z^\ell$, where $\ell = \ell_{i,j}$ denotes the length of the cycle $c_{i,j}$, $j = 1, \ldots, m_i$.

By "closing the puncture" of each $D_{i,j}^*$ and extending $f$ by continuity to the added point one obtains a compact Riemann surface $X$ and a ramified covering $f : X \to P^1(\mathbb{C})$ with the prescribed monodromy data.

**b)** Show that $X$ is connected iff the subgroup $G$ of $S_n$ generated by $\sigma_1 \ldots, \sigma_r$ acts transitively on $\{1, \ldots, n\}$.

**c)** If $X$ is connected, translate the Riemann-Hurwitz formula into a relation between the genus of $X$, the number $r$ of permutations and $m_1, \ldots, m_r$ of their cycles.

**d)** Check your formula on the polynomial exemples of previous exercices.

3. this results from Van Kampen theorem in covering theory.