Riemann surfaces

Final examination (duration : 3 hours)

The use of the course notes is allowed (but not the exercise sessions).

Exercise 1

Let us consider the Riemann surface $E = \mathbf{C}/\Lambda$ associated to the lattice $\Lambda = \mathbf{Z} + i\mathbf{Z}$ in $\mathbf{C}$.

1. Let $\phi : E \to E$ be a holomorphic map satisfying $\phi(0) = 0$. Show that there exists a unique $\alpha \in \mathbf{Z}[i]$ such that $\phi([z]) = [\alpha z]$ for every $z \in \mathbf{C}$.
2. Let G be the group of automorphisms ϕ of E satisfying $\phi(0) = 0$. Show that G is isomorphic to $\mu_4 = \{z \in \mathbf{C}^\times : z^4 = 1\}$.
3. Show that the natural action of G on E is faithful and proper. Is this action free ?
4. Determine all the points P of E whose stabilizer $\mathrm{Stab}(P) = \{g \in G : g(P) = P\}$ is non-trivial. For each such point, determine its stabilizer in G.
5. Determine the genus of the quotient Riemann surface E/G.
   We recall the Riemann-Hurwitz formula : if X and Y are compact connected Riemann surfaces of respective genera $g(X)$ and $g(Y)$, and if $\varphi : X \to Y$ is a non-constant holomorphic map, then
   $$2g(X) - 2 = (\deg \varphi)(2g(Y) - 2) + \sum_{P \in X} (e_\varphi(P) - 1).$$

The group G acts on the field of elliptic functions $\mathbf{C}(E) = \mathbf{C}(\Lambda)$ by the rule

$$g \cdot f = f \circ g \qquad (f \in \mathbf{C}(E), g \in G).$$

For any $\alpha \in \mu_4$, we will write $\alpha^* f = g_\alpha \cdot f$ where $g_\alpha \in G$ corresponds to α.

6. Determine the elliptic functions $i^*\wp_\Lambda$ and $i^*\wp'_\Lambda$.
7. Determine an explicit elliptic function $f \in \mathbf{C}(E)$ such that the fixed field $\mathbf{C}(E)^G$ is equal to $\mathbf{C}(f)$. Is this result consistent with the result of question 5 ?
8. Make explicit the Riemann surface E/G and the canonical projection map $\pi : E \to E/G$.

Exercise 2

Let X be a compact connected Riemann surface. Fix a point $P \in X$ and a finite set $S \subset X$ not containing P. The aim of this exercise is to show that there exists a meromorphic function f on X with a simple zero at P and no zeroes or poles at any of the points of S.

1. Show that the result holds for the Riemann sphere $X = \mathbf{P}^1(\mathbf{C})$.
2. Show that the result holds for an arbitrary complex torus $X = \mathbf{C}/\Lambda$.
3. Show that the result holds if X has genus $g \geq 2$.
   You may freely use the Riemann-Roch theorem.

Exercise 3

1. Compute the divisor of the meromorphic differential form $\omega = \frac{dz}{z}$ on $\mathbf{P}^1(\mathbf{C})$.
2. Let $P \in \mathbf{C}[T]$ be a non-constant polynomial. Compute in terms of P the residue at $z = \infty$ of the meromorphic differential form $\omega = \frac{P'(z)}{P(z)} dz$ on $\mathbf{P}^1(\mathbf{C})$.
3. Let $(r_a)_{a \in \mathbf{P}^1(\mathbf{C})}$ be a family of complex numbers indexed by $\mathbf{P}^1(\mathbf{C})$ such that $r_a = 0$ for all but finitely many a, and such that $\sum_{a \in \mathbf{P}^1(\mathbf{C})} r_a = 0$. Show that there exists a meromorphic differential form ω on $\mathbf{P}^1(\mathbf{C})$ having at most simple poles, and such that for every $a \in \mathbf{P}^1(\mathbf{C})$, we have $\mathrm{Res}_{z=a}(\omega) = r_a$.

Exercise 4

Let X be a compact connected Riemann surface. Show that if X admits an open set U biholomorphic to $\mathbf{C}$, then X is isomorphic to $\mathbf{P}^1(\mathbf{C})$.

You may freely use the uniformization theorem.