Riemann surfaces

Partial examination (duration : 2 hours)

Exercise 1

Let us consider the map

$$\begin{aligned} f : \mathbf{P}^1(\mathbf{C}) &\to \mathbf{P}^1(\mathbf{C}) \\ (x : y) &\mapsto (x^3(x-y)^2 : y^5). \end{aligned}$$

(1) Show that f is well-defined and holomorphic.

(2) Compute the ramification points of f and the associated ramification indices. What is the degree of f ?

Exercise 2

Let X be a compact connected Riemann surface, and let S be a finite subset of X.

(1) Let $f : X \backslash S \to \mathbf{C}$ be a bounded holomorphic function. Show that f is constant.

Let Y be a compact connected Riemann surface, and let $f : X\backslash S \to Y$ be a non-constant holomorphic map.

(2) Show that the image of f comes arbitrarily close to every point $Q \in Y$.
You may admit the following fact : there exists a non-constant meromorphic function g on Y whose only pole is at Q.

(3) Does f necessarily extend to a holomorphic map $\hat{f} : X \to Y$?

(4) Let S' be a finite subset of Y. Prove that any biholomorphic map $f : X\backslash S \to Y\backslash S'$ extends to a biholomorphic map $\hat{f} : X \to Y$.

Exercise 3

Let X be a compact connected Riemann surface. For any divisor $D = \sum_{P\in X} n_P[P]$ on X, we denote by $\mathrm{Supp}(D) = \{P \in X : n_P \neq 0\}$ the support of D.

For any nonzero meromorphic function $f \in \mathcal{M}(X)^\times$ such that $\mathrm{div}(f)$ and D have disjoint supports, we define

$$f(D) := \prod_{P\in \mathrm{Supp}(D)} f(P)^{n_P} \in \mathbf{C}^\times.$$

The aim of this exercise is to prove *Weil's reciprocity law* : for any nonzero meromorphic functions $f, g \in \mathcal{M}(X)^\times$ such that $\mathrm{div}(f)$ and $\mathrm{div}(g)$ have disjoint supports, we have

$$f(\mathrm{div}(g)) = g(\mathrm{div}(f)).$$

(1) Prove Weil's reciprocity law when f or g is a constant function.

(2) Assume $X = \mathbf{P}^1(\mathbf{C})$. Let a, b, c, d be four distinct points of $\mathbf{C}$. Verify Weil's reciprocity law when $f(z) = (z-a)/(z-b)$ and $g(z) = (z-c)/(z-d)$.

(3) Prove Weil's reciprocity law in the case $X = \mathbf{P}^1(\mathbf{C})$.

Let $\varphi : X \to Y$ be a non-constant holomorphic map between compact connected Riemann surfaces. We define $\varphi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ to be the unique $\mathbf{Z}$-linear map satisfying

$$\varphi^*[Q] = \sum_{P \in \varphi^{-1}(Q)} e_\varphi(P)[P] \qquad (Q \in Y).$$

(4) Show that $\varphi^*(\mathrm{div}\, h) = \mathrm{div}(\varphi^* h)$ for all $h \in \mathcal{M}(Y)^\times$.

We define $\varphi_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ to be the unique $\mathbf{Z}$-linear map satisfying $\varphi_*([P]) = [\varphi(P)]$ for all $P \in X$. We further define $\varphi_* : \mathcal{M}(X)^\times \to \mathcal{M}(Y)^\times$ to be the norm map associated to the field extension $\varphi^* : \mathcal{M}(Y) \to \mathcal{M}(X)$. We admit the following formulas

$$\varphi_*(\mathrm{div}\, f) = \mathrm{div}(\varphi_* f)$$
$$f(\varphi^* D) = (\varphi_* f)(D)$$

for $f \in \mathcal{M}(X)^\times$ and $D \in \mathrm{Div}(Y)$ where both sides are defined.

(5) Prove Weil's reciprocity law for arbitrary X by using the map $\hat{g} : X \to \mathbf{P}^1(\mathbf{C})$ to reduce to (3).