Exercise 1 Let Λ be a lattice in $\mathbb{C}$, $k \geq 3$ an integer, and $G_k = \sum_{\lambda \in \Lambda \smallsetminus \{0\}} 1/\lambda^k$. Let also $\wp$ denote the Weierstrass function associated to Λ.

a) Show that the series defining G_k converges and is zero for k odd.

b) Show that the Laurent expansion of $\wp$ near 0 is

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + O(z^6).$$

c) Deduce that

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + O(z^2)$$

and

$$\wp(z)^3 = \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + O(z^2).$$

d) Conclude that $\wp$ satisfies the Weierstrass differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6.$$

Exercise 2 Keep the notations of previous exercise.

a) Show that if $z_0 \in \mathbb{C} \smallsetminus \Lambda$, $\mathrm{div}(\wp - \wp(z_0)) = [P_0] + [-P_0] - 2[0]$, where $P_0 = z_0 + \Lambda \in \mathbb{C}/\Lambda$.

b) Let $f \in \mathbb{C}(\Lambda)^*$ be an even nonzero Λ-elliptic function. Show that its divisor of zeros and poles is of the form

$$\mathrm{div}(f) = \sum_{1 \leq i \leq k} n_i([P_i] + [-P_i])$$

with $n_i \in \mathbb{Z}$ and $P_i = z_i + \Lambda \in \mathbb{C}/\Lambda$ (observe that $z \mapsto f(z+\omega)$ is even for $\omega \in \frac{1}{2}\Lambda$).

c) Show that $\sum_i n_i = 0$, and deduce that the function

$$g = \prod_{1 \leq i \leq k, z_i \notin \Lambda} (\wp - \wp(z_i))^{n_i}$$

has the same divisor of zeros and poles as f.

d) Deduce that $f = cg$ for $c \in \mathbb{C}^*$, and that f belongs to $\mathbb{C}(\wp)$, the subfield of $\mathbb{C}(\Lambda)$ generated by $\wp$.

e) Show that $\mathbb{C}(\Lambda) = \mathbb{C}(\wp) \oplus \mathbb{C}(\wp)\wp' = \mathbb{C}(\wp, \wp')$, a field extension of degree 2 of $\mathbb{C}(\wp)$.

Exercise 3 Still keep the notations of previous exercises. Let

$$\phi : \mathbb{C}/\Lambda \smallsetminus \{0\} \to \mathbb{C}^2$$

$$z + \Lambda \mapsto (\wp(z), \wp'(z)).$$

a) Let $E \subset \mathbb{C}^2$ defined by the equation $y^2 = 4x^3 - ax - b$, where $a = 60G_4$, $b = 140G_6$. Show that $\phi(\mathbb{C}/\Lambda \smallsetminus \{0\}) = E$.

b) Show that ϕ is injective (consider $\mathrm{div}(\wp - x)$).

c) Let (ω_1, ω_2) be a $\mathbb{Z}$-basis of Λ. Show that the roots of $4x^3 - ax - b = 0$ are $\wp(\omega_1/2)$, $\wp(\omega_2/2)$, $\wp((\omega_1+\omega_2)/2)$ (consider $\mathrm{div}(\wp')$ and the parity of $\wp'$). Deduce that they are distinct.

d) Show that E is a complex submanifold of $\mathbb{C}^2$.

e) Conclude that ϕ is a biholomorphism from $\mathbb{C}/\Lambda \smallsetminus \{0\}$ to E.

f) What happens near $0 \in \mathbb{C}/\Lambda$?

Exercise 4 Let $\mathbb{S}^2$ be the unit sphere $\{(u,t) \in \mathbb{C} \times \mathbb{R} \mid |u|^2 + t^2 = 1\}$, and $\pi : \mathbb{S}^2 \setminus \{(0,1)\} \to \mathbb{C}$ the stereographic projection.

a) Check that $\pi(u,t) = u/(1-t)$, and that if $z = \pi(u,t)$, $|z|^2 + 1 = 2/(1-t)$.

b) Compute the riemannian metric $|du|^2 + dt^2$ on the sphere in terms of z and dz, assuming $t \neq 1$ (Let $z = \pi(u,t)$. Express first du in terms of z, t, dz, dt. Show that $\mathrm{Re}(\overline{z}dz) = dt/(1-t)^2$. Then simplify $|du|^2 + dt^2$). You should find an expression of the form $a(z)|dz|^2$ for some positive function a, showing that π is a conformal map, meaning that its tangent map preserves orthogonality of tangent vectors.

c) Show that images by π of circles on $\mathbb{S}^2$ (with $(0,1)$ removed if necessary) are circles or lines in $\mathbb{C}$.

Exercise 5 Let $H = \{(u,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0,\ t^2 - |u|^2 = 1\}$, and consider the projection $\pi : H \to \mathbb{C}$ from the point $(0,-1) \in \mathbb{C} \times \mathbb{R}$.

a) Check that $\pi(u,t) = u/(1+t)$, and that if $z = \pi(u,t)$, $1 - |z|^2 = 2/(1+t)$. What is the image of π ? Check that π is a diffeomorphism from H onto its image.

b) Compute the quadratic differential form $dt^2 - |du|^2$ on H in terms of z and dz (proceed as in the previous exercise). You should again find a metric of the form $a(z)|dz|^2$ on $\pi(H)$.

c) What are the images by π of the intersections of H with planes containing the origin ?