Exercise 1

a) Let $f : X \to Y$ be a nonconstant holomorphic map between connected Riemann surfaces. For a point $p \in X$, let $q = f(p) \in Y$, and choose local holomorphic coordinates z, w centered at p and q respectively, meaning $z(p) = 0$, $w(q) = 0$.

Show that there is an integer $m \geq 1$ such that $w \circ f = az^m$ in a neighborhood of p with a holomorphic and $a(p) \neq 0$, and that m doesn't depend on the choice of coordinates. The integer $m = m_f(p)$ might be called the "multiplicity" of f at p (or of p in $f^{-1}(q)$).

b) Show that the set of points p with $m_f(p) > 1$ (the "critical points" of f) is a closed discrete subset of X, hence finite if X is compact.

c) With the notations of the first question, show that one can choose the coordinate z so that $w \circ f = z^m$ near p.

d) Show that if f is bijective it is a biholomorphism, i.e. that f^{-1} is holomorphic. What can one conclude if f is only injective ?

e) Let $f : X \to Y$ be a nonconstant holomorphic map between compact connected Riemann surfaces. Show that for any point q in Y, the set $f^{-1}(q)$ is finite, and that the sum of multiplicities $\sum_{p \in f^{-1}(q)} m_f(p)$ doesn't depend on q (show that the sum is a locally constant function of q).

Exercise 2 A meromorphic function $f : X \dashrightarrow \mathbb{C}$ on a Riemann surface X is a holomorphic function $X \smallsetminus S \to \mathbb{C}$ for some closed discrete subset S of X such that f has (at most) a pole at each point $p \in S$. Two such functions are identified if they agree on some $X \smallsetminus S$ with S as above. Extending f continuously, it is the same thing as a holomorphic map $f : X \to P^1(\mathbb{C}) = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, not identically ∞ on any connected component of X.

a) Show that any rational fraction $F = P/Q \in \mathbb{C}(T)$ defines a meromorphic function f on $P^1(\mathbb{C})$, which conversely determines F (and is usually identified to it). The goal of the exercise is to show that all meromorphic functions on $P^1(\mathbb{C})$ are of this form.

b) Let $f : P^1(\mathbb{C}) \dashrightarrow \mathbb{C}$ be a meromorphic function on the Riemann sphere. Show the existence of a non-zero polynomial function $h : \mathbb{C} \to \mathbb{C}$ such that $g(z) = h(z)f(z)$ defines a meromorphic function on $P^1(\mathbb{C})$, without poles in $\mathbb{C}$.

c) If ∞ is a pole of order $m \geq 0$ of g, show that g is a polynomial function of degree m on $\mathbb{C}$.

d) Conclude that any meromorphic function on $P^1(\mathbb{C})$ is of the form $f(z) = P(z)/Q(z)$ for a rational fraction $F = P/Q \in \mathbb{C}(T)$.

e) Let $F \in \mathbb{C}(T)$ be a rational fraction, expressed as $F = P/Q$ for polynomials $P, Q \in \mathbb{C}[T]$ without common zero. Show that for all $t \in P^1(\mathbb{C})$ except a finite number, $f^{-1}(t)$ has cardinal $d = \max(\deg(P), \deg(Q))$ (use exercise 1).

f) Deduce that the automorphisms (biholomorphisms) of the Riemann sphere $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ are the homographies $z \mapsto f(z) = (az + b)/(cz + d)$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ ("Möbius transformations"). They constitute the *projective linear group* of invertible complex 2 by 2 matrices up to scalars

$$\mathrm{PGL}(2, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* \simeq \mathrm{Aut}(P^1(\mathbb{C})) \ .$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \longleftrightarrow \quad z \mapsto \frac{az + b}{cz + d} \ .$$

g) Show that for any three disctinct points $z_1, z_2, z_3 \in P^1(\mathbb{C})$, there is a unique $f \in \mathrm{PGL}(2, \mathbb{C})$ sending respectively z_1, z_2, z_3 to $0, 1, \infty$.

Exercise 3 Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with only simple zeros.

a) Show that $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = f(x)\}$ is a (non-compact) Riemann surface. What happens if f has multiple zeros ?

b) Show that $h : X \to \mathbb{C}$, defined by $h(x, y) = x$, is holomorphic, and proper.

c) Show that the set of $p \in X$ with $m_h(p) > 1$ (exercise 1) is $f^{-1}(0) \times \{0\}$, and that $m_h(p) = 2$.

d) Assume that f is a polynomial of degree $m \geq 1$, and consider $X_R = h^{-1}(\mathbb{C} \setminus D(0, R))$ for large R — a neighborhood of infinity in X.

Show that X_R is biholomorphic to $W = \{(u, v) \in U \times \mathbb{C}; v^2 = u^m\} \setminus \{(0, 0\}$ with U a connected neighborhood of 0 in $\mathbb{C}$ (start with coordinates $u_1 = 1/x$, $v = 1/y$).

e) Deduce that X_R is connected if m is odd, and disconnected if m is even.

f) Conclude that X can be embedded in a compact Riemann surface $\widehat{X}$, with $\widehat{X} \setminus X$ consisting of one point if m is odd, and two points otherwise (if m is odd, show that for $r > 0$ small enough, $W' = W \cap \{|u| < r\}$ is biholomorphic to the punctured disc $D(0, \sqrt{r}) \setminus \{0\}$ via $z \mapsto (z^2, z^m)$).