**Exercise 1**

**a)** Determine the ramification points, ramification indices and branch points of the maps $P^1(\mathbb{C}) \to P^1(\mathbb{C})$ given by

$$z \mapsto z^2 + 1, \quad z \mapsto z + \frac{1}{z}, \quad z \mapsto z^3/(1 - z^2).$$

**b)** Considering these as meromorphic functions on $P^1(\mathbb{C})$, determine their zeroes and poles, and the values of $\mathrm{ord}_p$ there.

**c)** Let $\Lambda$ be a lattice in $\mathbb{C}$, and $\wp$ the associated Weierstrass function, considered as a meromorphic function $\wp : \mathbb{C}/\Lambda = E \to P^1(\mathbb{C})$. Determine the ramification points, ramification indices and branch locus of $\wp$ (the set of branch points).

**d)** Construct an entire function $f : \mathbb{C} \to \mathbb{C}$ such that its branch locus is not closed in $\mathbb{C}$ (hint : perturb $\cos(z)$ and compose with another function).

**Exercise 2** (from previous session)

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with only simple zeros.

**a)** Show that $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = f(x)\}$ is a (non-compact) Riemann surface. What happens if $f$ has multiple zeros ?

**b)** Show that $h : X \to \mathbb{C}$, defined by $h(x, y) = x$, is holomorphic, and proper.

**c)** Show that the set of points $p \in X$ with ramification index $e_h(p) > 1$ is $f^{-1}(0) \times \{0\}$, and that $e_h(p) = 2$ for those points. What is the branch locus of $h$ ?

**d)** Assume from now on that $f$ is a polynomial of degree $m \geq 1$, and consider $X_R = h^{-1}(\mathbb{C} \smallsetminus D(0, R))$ for large $R$ — a neighborhood of infinity in $X$.

Show that $X_R$ is biholomorphic to $W = \{(u, v) \in U \times \mathbb{C}; v^2 = u^m\} \smallsetminus \{(0, 0\}$ with $U$ a connected neighborhood of 0 in $\mathbb{C}$ (start with coordinates $u_1 = 1/x$, $v = 1/y$).

**e)** Deduce that $X_R$ is connected if $m$ is odd, and disconnected if $m$ is even.

**f)** Conclude that $X$ can be embedded in a compact Riemann surface $\widehat{X}$, with $\widehat{X} \smallsetminus X$ consisting of one point if $m$ is odd, and two points otherwise (if $m$ is odd, show that for $r > 0$ small enough, $W' = W \cap \{|u| < r\}$ is biholomorphic to the punctured disc $D(0, \sqrt{r}) \smallsetminus \{0\}$ via $z \mapsto (z^2, z^m)$).

**Exercise 3**

**a)** Let $f : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ be a polynomial function of degree $d > 0$. Show that

$$\sum_{p \in P^1(\mathbb{C})} (e_f(p) - 1) = 2d - 2.$$

**b)** Let $X, Y, Z$ be compact connected Riemann surfaces, and $f : X \to Y$, $g : Y \to Z$ non constant holomorphic maps. show that for $p \in X$, $e_{g\circ f}(p) = e_g(f(p))e_f(p)$. In particular if $g$ is biholomorphic $e_{g\circ f} = e_f$.

**c)** Let $f : P^1(\mathbb{C}) \to P^1(\mathbb{C})$ be given by a non-constant irreducible rational fraction $f(z) = P(z)/Q(z)$, for complex polynomials $P$, $Q$. Let $d = \max(\deg P, \deg Q)$ be the degree of $f$.

Assuming that $\deg P \neq \deg Q$, show that

$$\sum_{p \in P^1(\mathbb{C})} (e_f(p) - 1) = 2d - 2.$$

**d)** Deduce the same equality if $\deg P = \deg Q$ (hint : compose with a translation, and use b)).