Corrigé de l'exercice 2.

Exercise 2.
Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with only simple zeros.

a) *Show that $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = f(x)\}$ is a (non-compact) Riemann surface. What happens if f has multiple zeros ?*

Let (x_0, y_0) be a point of X. Then either
(1) $y_0 = 0$, $f(x_0) = 0$ hence $f'(x_0) \neq 0$ since zeros are simple. Then there exists a (holomorphic) local inverse ϕ of f such that $\phi(0) = x_0$, and in a neighbourhood V of (x_0, y_0), X is defined by the equation $x = \phi(y^2)$. Choose y as coordinate on $V \cap X$.
or
(2) $y_0 \neq 0$, then there exists locally a (holomorphic) square root g of f such that $g(x_0) = y_0$. In a neighbourhood U of (x_0, y_0), X is defined by the equation $y = g(x)$. Choose x as coordinate on $U \cap X$.
Then X is covered by local coordinate charts, with coordinates either x or y, and in the intersections the coordinate change are holomorphic, given by $y = g(x)$, $x = \phi(y^2)$, or the identity.
This endows X with a structure of Riemann surface, such that the embedding $X \subset \mathbb{C}^2$ is holomorphic (both coordinates are holomorphic functions on X).
Alternatively, one could use the holomorphic local inversion theorem (see [1,chap. X], or [2, chap.4]) applied to $F(x, y) = f(x) - y^2$. The function $F : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic (complex differentiable in two variables), and if $F(x_0, y_0) = 0$, $\partial_1 F(x_0, y_0) = f'(x_0)$, $\partial_2 F(x_0, y_0) = -2y_0$ are not both zero. Then near (x_0, y_0), either $(F(x, y), y)$ or $(x, F(x, y))$ is a local biholomorphism of $\mathbb{C}^2$ mapping X to one of the coordinate axes, making X a complex submanifold of $\mathbb{C}^2$.
If f has multiple zeros, say $f(x) = c(x)(x - x_0)^m$ near x_0 with $c(x_0) \neq 0$, $m \geq 2$, $u = c(x)^{1/m}(x - x_0)$ is a local holomorphic coordinate on the x axis near x_0. Then X has local equation $y^2 = u^m$, and neither x nor y is locally injective near the point $(u, y) = (0, 0)$, so that X is not a smooth complex submanifold of $\mathbb{C}^2$.

b) *Show that $h : X \to \mathbb{C}$, defined by $h(x, y) = x$, is holomorphic, and proper.*

This is clear from the previous discussion.

c) *Show that the set of points $p \in X$ with ramification index $e_h(p) > 1$ is $f^{-1}(0) \times \{0\}$, and that $e_h(p) = 2$ for those points. What is the branch locus of h ?*

Let $p = (x_0, y_0)$ be a point of X. If $y_0 \neq 0$, $h = x$ is a local coordinate on X near p so that $e_h(p) = 1$. If $y_0 = 0$, y is a local coordinate on X near p and $x = \phi(y^2)$ for ϕ a local inverse of f such that $\phi(0) = x_0$. Hence $e_h(p) = 2$, since f is a centered local coordinate on C near x_0 and $f \circ h = y^2$ near p (i.e. $y = 0$).

d) *Assume from now on that f is a polynomial of degree $m \geq 1$, and consider $X_R = h^{-1}(\mathbb{C} \smallsetminus D(0, R))$ for large R — a neighborhood of infinity in X.*
Show that X_R is biholomorphic to $W = \{(u, v) \in U \times \mathbb{C}; v^2 = u^m\} \smallsetminus \{(0, 0\}$ with U a connected neighborhood of 0 in $\mathbb{C}$ (start with coordinates $u_1 = 1/x$, $v = 1/y$).

Write $f(x) = f^*(1/x)x^m$, where f^* is the reciprocal polynomial of f, of degree $\leq m$ and with $f^*(0) \neq 0$. If R is large enough, one has $f^*(1/x) = a(1/x)^m$ for $|x| > R$ and a holomorphic and non zero in a neighborhood of 0 in $\mathbb{C}$. Then $(x, y) \in X_R$ if and only if $|x| > R$ and $y^2 = (a(1/x)x)^m$. Choosing $u = 1/(xa(1/x))$ and $v = 1/y$ as coordinates near infinity in $\mathbb{C}$, one sees that X_R is biholomorphic to W, where $U \setminus \{0\}$ is the image of $\{x; |x| > R\}$ by $x \mapsto 1/(xa(1/x))$ (a punctured disk).

***e)** Deduce that X_R is connected if m is odd, and disconnected if m is even.*

If m is even, W has two connected components given by $v = \pm u^{m/2}$, $u \in U \setminus \{0\}$.
If m is odd and $(u, v) \in W$, let z be a square root of u. Then $v^2 = z^{2m}$, hence $v = \pm z^m$. Changing z to $-z$ if necessary, one has $u = z^2$ and $v = z^m$, and such a z is unique. Thus $z \mapsto (z^2, z^m)$ is a bijective holomorphic map

$$\psi : U' \setminus \{0\} \to W$$

from a punctured neighborhood $U' \setminus \{0\}$ of 0 in $\mathbb{C}$ to W, where U' is the inverse image of U by $z \mapsto z^2$, hence still a punctured disk. In particular W is connected. Note that, being a holomorphic bijection between Riemann surfaces, ψ is a biholomorphism.

***f)** Conclude that X can be embedded in a compact Riemann surface $\widehat{X}$, with $\widehat{X} \setminus X$ consisting of one point if m is odd, and two points otherwise (if m is odd, show that for $r > 0$ small enough, $W' = W \cap \{|u| < r\}$ is biholomorphic to the punctured disc $D(0, \sqrt{r}) \setminus \{0\}$ via $z \mapsto (z^2, z^m)$).*

- If the degree m of the polynomial f is even, we know that $X_R \simeq W$ is biholomorphic to the union of the two images by $\psi^\pm : u \mapsto (u, \pm u^{m/2})$ of the punctured disk $U \setminus \{0\}$.
By adding two distinct points $O^\pm$ to X and setting $\psi^\pm(0) = O^\pm$, $U^\pm = \psi^\pm(U)$ one defines a compact Riemann surface $\widehat{X} = X \sqcup \{O^+, O^-\}$, with $(\psi^\pm)^{-1}$ as coordinates in $U^\pm$.
- If m is odd, W (hence X_R) is biholomorphic to the punctured disk $U' \setminus \{0\}$ by $\psi(z) = (z^2, z^m)$, and one can add a unique point O to X, and define $\psi(0) = O$ to obtain a compact Riemann surface $\widehat{X} = X \sqcup \{O\}$, with ψ^{-1} as holomorphic coordinate in $\psi(U')$.
The first coordinate map $h : X \to \mathbb{C}$ to the horizontal axis extends to a map $h : \widehat{X} \to P^1(\mathbb{C})$ in both case by sending the added point(s) to ∞. It is a holomorphic map between compact connected Riemann surface.
Indeed if m is even h is given in coordinates $u = (\psi^\pm)^{-1}$ in $U^\pm \subset \widehat{X}$ and $u = 1/x$ in $\mathbb{P}^1(\mathbb{C})$ by the identity, and $e_h(O^\pm) = 1$. If m is odd, h is given by $u = z^2$, with $z = \psi^{-1}$ the above coordinate centered in O on X. Hence $e_h(O) = 2$ if m is odd.
In both cases, $h : \widehat{X} \to P^1(\mathbb{C})$ is a map of degree 2, ramified in an even number of points (m or $m+1$), with branch locus $f^{-1}(0)$ if m is even and $f^{-1}(0) \cup \{\infty\}$ if m is odd. We will learn later that the compact Riemann surface $\widehat{X}$ is of genus $m/2 - 1$ resp. $(m-1)/2$. It is called a hyperelliptic (complex) curve.

[1] Jean Dieudonné, Eléments d'analyse, Tome 1, Editions Jacques Gabay, 1990.
[2] Henri Cartan, Calcul différentiel, Hermann, 1967.