Corrigé de l'exercice 3.

Exercise 3. Let $E = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$. Remember that Weierstrass function $\wp \in \mathcal{M}(E)$ is even and has a single pole at 0, which is of order 2. Also recall that every meromorphic function $f \in \mathcal{M}(E)$ may be written uniquely as $f = f_1 + f_2\wp'$ with f_1, f_2 rational functions of $\wp$.

a) *Show that if $f \in \mathcal{M}(E)^*$ is a rational function of $\wp$, $\mathrm{ord}_0(f)$ is even.*

Using that $\mathrm{ord}_p(g \circ f) = \mathrm{ord}_{f(p)}(g)e_f(p)$, we find $\mathrm{ord}_0(\phi \circ \wp) = \mathrm{ord}_{\wp(0)}(\phi)e_\wp(0)$, and $e_\wp(0) = 2$ since $\wp(z) = z^{-2} + O(1)$ near $z = 0$.

b) *For $c \in \mathbb{C}$ show that $\wp^{-1}(c)$ is not empty.*

Since $\mathrm{ord}_0(\wp) = -2$ and $\mathrm{div}(\wp - c)$ is of degree 0, we must have $\mathrm{div}(\wp - c) = -2[0] + [q_1] + [q_2]$ for some q_i's in E (we even know that in fact $q_2 = -q_1$), in particular $\wp(q_i) = c$.

c) *Show that if $\phi \in \mathbb{C}(T)$ is not a polynomial, $f = \phi \circ \wp$ has a pole in $E \smallsetminus \{0\}$.*

If $c \in \mathbb{C}$ is a pole of ϕ and $q \in \wp^{-1}(c)$, $\phi \circ \wp$ has a pole in q, and $q \in E \smallsetminus \{0\}$.

d) *Deduce that the vector space $L([0])$ contains only the constant functions.*

Let $f \in L([0])$, and write $f = f_1 + f_2\wp'$, with $f_i = \phi_i \circ \wp$, $\phi_i \in \mathbb{C}(T)$, $i = 1, 2$. Since $\mathrm{ord}_0(f_i)$ are even and $\mathrm{ord}_0(\wp') = -3$ is odd, we must have

$$\mathrm{ord}_0 f = \min(\mathrm{ord}_0(f_1), \mathrm{ord}_0(f_2) - 3).$$

Hence $\mathrm{ord}_0 f \geq -1$ forces $\mathrm{ord}_0 f_1 \geq 0$ and $\mathrm{ord}_0 f_2 \geq 2$.
On the other hand f_1 is the even part of f i.e. $f_1(z) = (f(z) + f(-z))/2$, thus it is holomorphic in $E \smallsetminus \{0\}$ since $\mathrm{div}(f) \geq -[0]$. Hence f_1 is holomorphic everywhere, implying it is constant. Similarly f_2 is holomorphic in $E \smallsetminus \{0\}$, and since it vanishes in 0, $f_2 = 0$.
Conclusion : f is constant.

e) *Conclude the same for $L([q])$, $q \in E$ (use a translation on E).*

The map $f \mapsto f(\bullet + q)$ form $\mathcal{M}(E)$ to itself sends $L([q])$ to $L([0])$.

f) *Show that $L(2[0])$ is of dimension 2, with basis $(1, \wp)$.*

Arguing as is question d), one finds that $f = f_1 + f_2\wp' \in L(2[0])$ must verify $\mathrm{ord}_0(f_1) \geq -2$, $\mathrm{ord}_0(f_2) \geq 2$ and f_1, f_2 holomorphic in $E \smallsetminus \{0\}$. Hence $f_2 = 0$, and since $f_1 = \phi_1 \circ \wp$ has no pole outside of 0, ϕ_1 must be a polynomial (question c)), of degree at most 1 because $\mathrm{ord}_0 f_1 \geq -2$.
Thus $f = \phi_1 \circ \wp = a + b\wp$, $a, b \in \mathbb{C}$.

g) *Deduce that if q is a point of E, $L(2[q])$ has dimension 2.*

Like above use the map $f \mapsto f(\bullet + q)$ in $\mathcal{M}(E)$.

h) *Generalize to show that $\dim L(n[q]) = n$, $n \geq 2$ (first reduce to the case $q = 0$; write n as $2a + 3b$ for integers $a, b \geq 0$ and use Laurent expansion).*

Let $q = 0$, $n = 2a + 3b \geq 2$ $(a, b \geq 0)$, and $f \in L(n[0])$. Then $g = \wp^a\wp'^b$ is also in $L(n[0])$ and its order at 0 is exactly $-n$, whence $\mathrm{div}(f - cg) \geq -(n-1)[0]$ for some $c \in \mathbb{C}$, and we obtain $L(n[0]) = L((n-1)[0]) \oplus \mathbb{C}g$. By induction from $L([0]) = \mathbb{C}$, the result follows.

i) *If q_1, q_2 are distinct points of E, show that $\dim L([q_1] + [q_2]) = 2$ (reduce to the case $q_1 = -q_2$, and use $\mathrm{div}(\wp - \wp(q_1)) = [q_1] + [-q_1] - 2[0]$, plus exercise 2.b).*

$L([q_1] + [q_2])$ is sent to $L([q_1 - a] + [q_2 - a])$ by $f \mapsto f(\bullet + a)$ so choose $a \in E$ such that $2a = q_1 + q_2$ (there are four choices).

Assume now $q_1 + q_2 = 0$. We have $\operatorname{div}(\wp - \wp(q_1)) = [q_1] + [-q_1] - 2[0]$ (this follows from the evenness of $\wp$), so the difference $([q_1] + [-q_1]) - 2[0]$ is a principal divisor on E, and from exercice 2.b $L([q_1] + [-q_1]) \simeq L(2[0])$. The conclusion follows from $\dim L(2[0]) = 2$.

j) *Let $p, q \in E$. Show that there is an $r \in E$ such that $2r = q$ and $p + r \neq 0$. Deduce that*

$$f(z) = \frac{\wp(z-r) - \wp(p+r)}{(\wp(z) - \wp(p))(\wp(z-r) - \wp(r))}$$

defines a function $f \in \mathcal{M}(E)^*$ such that $\operatorname{div}(f) = [p+q] - [p] - [q] + [0]$.

There are four solutions r to $2r = q$ in E, so at least three with $r \neq -p$.
Using $\operatorname{div}(\wp - \wp(a)) = [a] + [-a] - 2[0]$ for $a \in E \smallsetminus \{0\}$, and introducing the group homomorphism $T_r : \operatorname{Div}(E) \to \operatorname{Div}(E)$ defined by $T_r([a]) = [a + r]$ $(a \in E)$, one finds

$$\begin{aligned}\operatorname{div}(f) &= T_r([p+r] + [-p-r] - 2[0]) - ([p] + [-p] - 2[0]) - T_r([r] + [-r] - 2[0])\\ &= [p+q] + [-p] - 2[r] - [p] - [-p] + 2[0] - [q] - [0] + 2[r] = [p+q] - [p] - [q] + [0].\end{aligned}$$

This means that the map

$$\begin{aligned} E &\to \operatorname{Div}(E)/\operatorname{Pr}(E)\\ p &\mapsto [p] - [0] \pmod{\operatorname{Pr}(E)}\end{aligned}$$

is a *group homomorphism.*

k) *Deduce that for $n \geq 1$ points $p_1, \dots, p_n$ in E, the divisor $[p_1] + \cdots + [p_n] - [p_1 + \cdots + p_n] - (n-1)[0]$ is principal. If $p_1 + \cdots + p_n = 0$, conclude that $\dim L([p_1] + \cdots + [p_n]) = n$. Show that this is still true if $p_1 + \cdots + p_n \neq 0$ (use a translation).*

From previous question we conclude that for $D \in \operatorname{Div}(E)$ and all $p, q \in E$,

$$L(D + [p] + [q]) \simeq L(D + [p+q] + [0]).$$

Thus for $n \geq 1$

$$L([p_1] + \cdots + [p_n]) \simeq L([p_1 + \cdots + p_n] + (n-1)[0]).$$

Moreover by translation $d := \dim L([p_1] + \cdots + [p_n]) = \dim L([p_1 + r] + \cdots + [p_n + r])$ and we can choose r so that $\sum_i (p_i + r) = 0$, so we obtain $d = \dim L(n[0]) = n$ as wanted.

l) *Show that if D is a divisor on E with $\deg(D) \geq 1$, the space $L(D)$ is of dimension $\deg D$ (show that there is a $p \in E$ with $D - n[p]$ principal, $n = \deg(D)$).*

Write $D = \sum_{i=1}^k n_i [p_i]$, and $n = \sum n_i \geq 1$. Then if $\sum_i n_i p_i = 0 \in E$, $\sum_i n_i([p_i] - [0])$ is a principal divisor as a consequence of question j), hence $L(D) \simeq L(n[0])$.
If the sum $\sum_i n_i p_i \in E$ is not 0, we can at least write it as np for some $p \in E$ since $n \neq 0$. Then $\sum_i n_i(p_i - p) = 0$ and $\sum_i n_i([p_i] - [p])$ is principal.