Exercise 1. Let f be a polynomial of degree $n = 2m \geq 2$ without multiple roots, and consider $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = f(x)\}$. Recall that X is a non-compact Riemann surface, with local coordinate x (resp. y) near $(x_0, y_0) \in X$ when $y_0 \neq 0$ (resp. $y_0 = 0$).

Recall also that X can be extended to a compact Riemann surface $\widehat{X}$ by adding two points "at infinity" $\infty_\pm$ to the two components $y = \pm x^m g(1/x)$, of $X \cap \{|x| > R\}$, with R large enough and $g(1/x)$ a holomorphic square root of $f^*(1/x) = f(x)/x^{2m}$ in $\{|x| > R\}$. Then $u_\pm = 1/x$ extends to a local coordinate centered in $\infty_\pm$, and X has equation $v = 1/y = \pm u_\pm^m / g(u_\pm)$ for $u_\pm \neq 0$, $|u_\pm| < 1/R$ (i.e in a punctured neighborhood of $\infty_\pm$).

a) Show that the expression $\omega = dx/y$ defines a holomorphic 1-form on $X \smallsetminus \mathbb{C} \times 0$ which extends to a holomorphic 1-form ω on all of X.

b) Show that ω extends to a holomorphic 1-form on $\widehat{X}$ (still denoted ω).

c) Determine the divisor of ω, i.e. the zeros of ω and their multiplicities. What is its degree ?

d) If $k \geq 0$ is an integer, when does $x^k\omega$ extend to a holomorphic 1-form on $\widehat{X}$?

e) Show that any holomorphic 1-form α on $\widehat{X}$ coincides on X with $P(x)\omega$ for some polynomial $P \in \mathbb{C}[x]$ (hint : first consider the divisor of the meromorphic function α/ω, then show that $\alpha + \sigma^*\alpha = 0$ for $\sigma(x, y) = (x, -y)$, by showing must vanish at the zeros of y on X).

f) Determine the dimension p of $\Omega^1(\widehat{X})$. What is the dimension of $\Omega^1(X)$?

Exercise 2.

a) Let $f : X \to Y$ be a holomorphic map of Riemann surfaces.
Show that the pullback map on (complex, smooth) 1-forms $f^* : \mathcal{A}^1(Y) \to \mathcal{A}^1(X)$ satisfies $f^*(\mathcal{A}^{1,0}(Y)) \subset \mathcal{A}^{1,0}(X)$ and $f^*(\mathcal{A}^{0,1}(Y)) \subset \mathcal{A}^{0,1}(X)$.

b) Show that $f^*(\Omega^1(Y)) \subset \Omega^1(X)$.

c) If $g : Y \to Z$ is also a holomorphic map of Riemann surfaces, show that $(g \circ f)^* = f^* \circ g^*$.

d) Let $\Lambda \subset \mathbb{C}$ be a lattice, $\mathbb{C}/\Lambda$ the corresponding elliptic curve. Denote by $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ the (holomorphic) quotient map. Show that π^* is injective on $\mathcal{A}(\mathbb{C}/\Lambda)$ with image in $\mathcal{A}(\mathbb{C})$ the subspace of 1-forms $a(z)dz + b(z)d\overline{z}$ with a, b smooth Λ-periodic functions on $\mathbb{C}$.

e) Denote by ω the 1-form on $\mathbb{C}/\Lambda$ such that $\pi^*\omega = dz$ (differential of the canonical coordinate $z : \mathbb{C} \to \mathbb{C}$). Show that $\Omega^1(\mathbb{C}/\Lambda) = \mathbb{C}\omega$.

f) Let Λ' be another lattice in $\mathbb{C}$, with corresponding π' and ω', and $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ any holomorphic map. Show that $f^*\omega' = \alpha\omega$ for a constant $\alpha \in \mathbb{C}$.

g) Show that if $h : \mathbb{C} \to \mathbb{C}/\Lambda'$ is a smooth map such that $h^*\omega' = 0$, h is constant.

h) In the situation of question f), show that for some constant $\beta \in \mathbb{C}$ the map $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \alpha z + \beta$ satisfies $\pi' \circ g = f \circ \pi$. Conclude that $\alpha\Lambda \subset \Lambda'$.

i) Show that $\mathbb{C}/\Lambda$, $\mathbb{C}/\Lambda'$ are isomorphic as Riemann surfaces if and only if $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^*$.

j) Conclude that every elliptic curve is isomorphic to $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ for some τ with $\mathrm{Im}(\tau) > 0$, and that (more importantly) E_τ, $E_{\tau'}$ are isomorphic if and only if $\tau' = (a\tau + b)/(c\tau + d)$ for integers a, b, c, d such that $ad - bc = 1$.

Exercise 3. Let $v = a(x,y)\partial_x + b(x,y)\partial_y$ be a (tangent) vector field in an open subset U of $\mathbb{R}^2$, identified to an open subset of $\mathbb{C}$ by $z = x + iy$. One says that v is holomorphic if for every holomorphic function f in an open subset U' of U, the fonction $vf = a\partial_x f + b\partial_y f$ is holomorphic in U'.

a) Show that v is holomorphic if and only if $a + ib$ is a holomorphic function of z.

b) Show that if v is holomorphic and α is a holomorphic 1-form on U, the function $\alpha(v)$ is holomorphic on U.

c) Show that if V is an open subset of $\mathbb{C}$ and $\phi : U \to V$ a biholomorphism, the image of v by ϕ is holomorphic in V.

d) Let X be a Riemann surface. By the above, one can define the notion of holomorphic vector field on X. If X is compact and admits a non-zero holomorphic 1-form (resp. vector field) with at least one zero, then 0 is the only holomorphic vector field (resp. 1-form) on X.

e) Show that if X is compact and admits a holomorphic vector field without zeros, $\Omega^1(X)$ is one dimensional.

f) Show that if v is a holomorphic vector field on a Riemann surface X, its flow ϕ_t at time t, defined on an open subset $U_t \subset X$, is a holomorphic map from U_t to X.