Corrigé de l'exercice 2.

Exercise 2.

a) *Let $f : X \to Y$ be a holomorphic map of Riemann surfaces.*
Show that the pullback map on (complex, smooth) 1-forms $f^* : \mathcal{A}^1(Y) \to \mathcal{A}^1(X)$ satisfies $f^*(\mathcal{A}^{1,0}(Y)) \subset \mathcal{A}^{1,0}(X)$ and $f^*(\mathcal{A}^{0,1}(Y)) \subset \mathcal{A}^{0,1}(X)$.

For $\omega \in \mathcal{A}^1(Y)$, $p \in X$, $(f^*\omega)_p$ is given by $\omega_{f(p)} \circ df(p) : T_pX \to \mathbb{C}$. Since $df(p)$ is $\mathbb{C}$-linear, $(f^*\omega)_p$ is $\mathbb{C}$-linear (resp. antilinear) when $\omega_{f(p)}$ is.

b) *Show that $f^*(\Omega^1(Y)) \subset \Omega^1(X)$.*

If ω is a holomorphic one form on X, ω is closed and of type $(1,0)$, and conversely. Thus $f^*\omega$ is closed, and f being holomorphic, $f^*\omega$ is also of type $(1,0)$ (alternatively, a 1-form on a Riemann surface is holomorphic if and only if it is everywhere locally $u\,dv$ for u and v (local) holomorphic functions, and this is preserved by holomorphic pullback : $(u \circ f)\,d(v \circ f)$ is also of this form).

c) *If $g : Y \to Z$ is also a holomorphic map of Riemann surfaces, show that $(g \circ f)^* = f^* \circ g^*$.*

This is the "chain rule", $T(g \circ f) = Tg \circ Tf$ (if 1-forms are viewed as (particular) functions on the tangent bundle $f^*\omega = \omega \circ Tf$). This doesn't need f, g to be holomorphic, differentiable maps would suffice.

d) *Let $\Lambda \subset \mathbb{C}$ be a lattice, $\mathbb{C}/\Lambda$ the corresponding elliptic curve. Denote by $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ the (holomorphic) quotient map. Show that π^* is injective on $\mathcal{A}(\mathbb{C}/\Lambda)$ with image in $\mathcal{A}^1(\mathbb{C})$ the subspace of 1-forms $a(z)dz + b(z)d\overline{z}$ with a, b smooth Λ-periodic functions on $\mathbb{C}$.*

The quotient map π is a local diffeomorphism (biholomorphism), thus $T\pi$ is surjective, hence the linear map π^* is injective. Every element of $\mathcal{A}^1(\mathbb{C})$ writes uniquely as $a(z)dz + b(z)d\overline{z}$ with a, b smooth (complex) functions on $\mathbb{C}$. Clearly, such a form is in the image of π^* if and only if it is invariant by the translations $z \mapsto z + \lambda$, $\lambda \in \Lambda$, i.e. a and b are Λ-periodic.

e) *Denote by ω the 1-form on $\mathbb{C}/\Lambda$ such that $\pi^*\omega = dz$ (differential of the canonical coordinate $z : \mathbb{C} \to \mathbb{C}$). Show that $\Omega^1(\mathbb{C}/\Lambda) = \mathbb{C}\omega$.*

ω is a non-vanishing holomorphic 1-form on $X = \mathbb{C}/\Lambda$, hence any other one is of the form $u\,\omega$ for a holomorphic *function* u on X. But X being compact, u must be constant.

f) *Let Λ' be another lattice in $\mathbb{C}$, with corresponding π' and ω', and $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ any holomorphic map. Show that $f^*\omega' = \alpha\omega$ for a constant $\alpha \in \mathbb{C}$.*

This results from the previous question and the fact that $f^*\omega'$ is a holomorphic 1-form on $\mathbb{C}/\Lambda$.

g) *Show that if $h : \mathbb{C} \to \mathbb{C}/\Lambda'$ is a smooth map such that $h^*\omega' = 0$, h is constant.*

Since $\omega'_p : T_p(\mathbb{C}/\Lambda') \to \mathbb{C}$ is an isomorphism for all p, the differential $dh(z)$ must be 0 for each $z \in \mathbb{C}$. By connectedness of $\mathbb{C}$, h is constant.

h) *In the situation of question f), show that for some constant $\beta \in \mathbb{C}$ the map $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \alpha z + \beta$ satisfies $\pi' \circ g = f \circ \pi$. Conclude that $\alpha\Lambda \subset \Lambda'$.*

Define $h : \mathbb{C} \to \mathbb{C}/\Lambda'$ by $h(z) = f(\pi(z)) - \pi'(\alpha z)$. Then $h^*\omega' = \pi^* f^* \omega' - \alpha dz = 0$, hence h is constant by previous question. Choose $\beta \in \mathbb{C}$ such that the value of h is $\pi'(\beta)$. Then $g(z) = \alpha z + \beta$ defines $g : \mathbb{C} \to \mathbb{C}$ such that $f \circ \pi = \pi' \circ g$.
In particular $z \in \Lambda$ implies $\pi(z) = \pi(0)$, so $g(z) - g(0) = \alpha z \in \ker \pi' = \Lambda'$.

i) *Show that $\mathbb{C}/\Lambda$, $\mathbb{C}/\Lambda'$ are isomorphic as Riemann surfaces if and only if $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^*$.*

By the previous questions, for an isomorphism f with inverse f^{-1}, $f^*\omega' = \alpha\omega$ and $(f^{-1})^*\omega = \beta\omega'$ with $\alpha\Lambda \subset \Lambda'$, $\beta\Lambda' \subset \Lambda$, and obviously $\alpha\beta = 1$, which forces equality in the inclusions. Conversely, if $\Lambda' = \alpha\Lambda$, $z \mapsto \alpha z$ induces and isomorphism $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$.

j) *Conclude that every elliptic curve is isomorphic to $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ for some τ with $\mathrm{Im}(\tau) > 0$, and that (more importantly) E_τ, $E_{\tau'}$ are isomorphic if and only if $\tau' = (a\tau + b)/(c\tau + d)$ for integers a, b, c, d such that $ad - bc = 1$.*

If ω_1, ω_2 is a basis of Λ over $\mathbb{Z}$, $\omega_1^{-1}\Lambda$ contains 1, and choosing $\tau = \pm\omega_2/\omega_1$ with $\mathrm{Im}\,\tau > 0$ ones obtains $\omega_1^{-1}\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, and $\mathbb{C}/\Lambda$ is isomorphic to $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.
$E_\tau, E_{\tau'}$ are isomorphic if and only if $\mathbb{Z} + \mathbb{Z}\tau = \alpha(\mathbb{Z} + \mathbb{Z}\tau')$ for some $\alpha \in \mathbb{C}^*$.
But this means that $(\alpha,\alpha\tau')$ is another $\mathbb{Z}$-basis of $\mathbb{Z} + \mathbb{Z}\tau$, so that $\alpha\tau' = a\tau + b$, $\alpha = c\tau + d$ for integers a, b, c, d with $ad - bc = \pm 1$. Then $\tau' = (a\tau + b)/(c\tau + d)$.
Computing the imaginary part shows that necessarily $ad - bc = 1$. Conversely this equality implies $\mathbb{Z} + \mathbb{Z}\tau = \alpha(\mathbb{Z} + \mathbb{Z}\tau')$ for $\alpha = c\tau + d$.