Exercise 1. Let X be a compact Riemann surface.

a) Using the residue theorem, give a proof that for any non-zero meromorphic f function on X, the degree of $\mathrm{div}(f)$ is zero (hint : use a meromorphic one form constructed from f).

b) Show that if f is a meromorphic function in a neighbourhood of $x \in X$, $\mathrm{res}_x(df) = 0$.

c) Deduce that for $x \in X$, the $\mathbb{C}$-bilinear form $(f, g) \mapsto \mathrm{res}_x(fdg)$ is antisymmetric on the vector space $\mathcal{M}_x$ of germs of meromorphic functions at x. What is its kernel ?

Exercise 2.

a) Let C be the cylinder (or annulus) $[0, 1] \times \mathbb{S}^1$, and ω a $\mathcal{C}^1$ closed one form on C. Using Stokes theorem, show that

$$\int_{0\times\mathbb{S}^1} \omega = \int_{1\times\mathbb{S}^1} \omega.$$

b) Deduce that if X is a Riemann surface, $\omega \in \Omega^1(X)$ a holomorphic form on X, and $\gamma_0, \gamma_1 : \mathbb{S}^1 \to X$ two $\mathcal{C}^1$ curves in X such that $\gamma_0 = h(0, \cdot)$, $\gamma_1 = h(1, \cdot)$ for a $\mathcal{C}^1$ map $h : [0, 1] \times \mathbb{S}^1 \to X$, then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

c) Deduce that if ω is a holomorphic one form in an open subset of $\mathbb{C}$ containing the square $S = [0, 1] \times [0, 1]$, one has $\int_{\partial S} \omega = 0$ (hint : parameterize ∂S by a $\mathcal{C}^1$ map from $\mathbb{S}^1$. Then use homotheties).

Exercise 3.

a) Let Λ be a lattice in $\mathbb{C}$. Show that a holomorphic map $f : \mathrm{P}^1(\mathbb{C}) \to \mathbb{C}/\Lambda$ is necessarily constant.

b) More generally, prove that if X is a connected Riemann surface which admits a non-zero holomorphic one form, any holomorphic map $f : \mathrm{P}^1(\mathbb{C}) \to X$ is constant.