Exercise 1. Let $N \geq 3$ and $P \in \mathbb{C}[x, y]$ be a degree N polynomial such that for $(x, y) \in \mathbb{C}^2$, $P(x, y) = 0$, implies $(\partial P(x, y)/\partial x, \partial P(x, y)/\partial y) \neq (0, 0)$, and moreover such that its homogeneous component of maximal degree is of the form $P_N(X, Y) = \prod_{1 \leq k \leq N}(Y - \alpha_k X)$ for distinct complex numbers, $\alpha_1, \ldots, \alpha_N$.

a) Verify that $P(x, y) = x^N + y^N - 1$ satisfies the hypotheses, but not $P(x, y) = y^2 - f(x)$ with f of degree N (assumed ≥ 3). In the following questions, you can first assume that $P = x^N + y^N - 1$ ($P^{-1}(0)$ is the affine "Fermat curve").

b) Show that $X = \{(x, y) \in \mathbb{C}^2; P(x, y) = 0\}$ is a (non-compact) Riemann surface, and that x (resp. y) is a local coordinate on X near points (x_0, y_0) such that $\partial P(x_0, y_0)/\partial y \neq 0$ (resp. $\partial P(x_0, y_0)/\partial x \neq 0$).

c) Abbreviate $P_x = \partial P/\partial x$ and $P_y = \partial P/\partial y$. Show that the expression $dx/P_y(x, y)$ defines a non-zero holomorphic 1-form on $X \cap \{P_y \neq 0\}$, as does $-dy/P_x(x, y)$ on $X \cap \{P_x \neq 0\}$. Verify that these two holomorphic 1-forms agree on the intersection of their domains of definition, and define a holomorphic 1-form ω on X.

d) Show that the change of variables $x = 1/u, y = v/u$, $u \in \mathbb{C}^*$, $v \in \mathbb{C}$ defines a biholomorphism of $\mathbb{C}^* \times \mathbb{C}$ which maps $X \cap \{x \neq 0\}$ to the curve defined by $P^*(u, v) = 0$ in $\mathbb{C}^* \times \mathbb{C}$, with

$$P^*(u, v) = u^N P(1/u, v/u) = P_N(1, v) + uP_{N-1}(1, v) + \cdots + u^N P_0.$$

e) Let $p_k(v) = P_k(1, v)$ for $k = 0, \ldots, N$, and recall that by hypothesis

$$p_N(v) = \prod_{1 \leq k \leq N} (v - \alpha_k)$$

with distinct roots $\alpha_1, \ldots, \alpha_N$. Show that the curve defined by $P^*(u, v) = 0$ in $\mathbb{C}^2$ is a Riemann surface intersecting the vertical axis $u = 0$ in the points $(0, \alpha_k)$, near which u is a local coordinate.

f) Deduce that X admits a compactification by N points $\widehat{X} = X \sqcup \{p_1, \ldots, p_N\}$ which is a Riemann surface.

g) Show that the holomorphic 1-form $\omega = dx/P_y = -dy/P_x$ of question c) extends as a holomorphic form on $\widehat{X}$, still denoted ω (recall that $N \geq 3$).

h) Compute the divisor of ω and its degree. What is the genus of $\widehat{X}$? Compute it for $N = 3, 4, 5$.

i) Show that if $F \in \mathbb{C}[x, y]$, $F(x, y)\omega$ defines a meromorphic form on $\widehat{X}$.

j) Show that the vector subspace of forms $F\omega$, ($F \in \mathbb{C}[x, y]$) which are holomorphic on $\widehat{X}$ is of dimension $(N - 1)(N - 2)/2$.

k) Deduce that these are *all* the holomorphic forms on $\widehat{X}$ (hint : use the degree of $\mathrm{div}(\omega)$ computed in question h).

Exercise 2.

a) Show that the quotient Riemann surface $\mathbb{C}/\mathbb{Z}$ (with $\mathbb{Z}$ acting naturally by translations) is isomorphic to $\mathbb{C}^*$ (hint : show that the map $\mathbf{e} : \mathbb{C} \to \mathbb{C}^*$, $\mathbf{e}(z) = \exp(2i\pi z)$ defines a biholomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$).

b) Let $H = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$. Show that the quotient Riemann surface $H/\mathbb{Z}$ (with $\mathbb{Z}$ acting naturally by translations) is isomorphic to the punctured unit disc $\mathbb{D}^* = \{q \in \mathbb{C}^*; |q| < 1\}$.

c) Let $\mu_n \subset \mathbb{C}^*$ be the group of n-th roots of 1, acting on $\mathbb{C}$ by multiplication. Show that the quotient Riemann surface $\mathbb{C}/\mu_n$ is isomorphic to $\mathbb{C}$.

d) Show that the action of $\mathbb{Z}$ on $\mathrm{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ generated by $z \mapsto 2z$ is not proper. Show that it is proper on the complement of the fixed points. What is the corresponding quotient ?

e) Same as the previous question, but for $z \mapsto z+1$ on $\mathrm{P}^1(\mathbb{C})$.

f) What about the action of μ_n (by multiplication) on $\mathrm{P}^1(\mathbb{C})$?

g) Consider the action of μ_N on the compact Fermat curve $\widehat{X}$ of exercice 1, given on its affine part X by $\lambda \cdot (x, y) = (\lambda x, y)$, $\lambda^N = 1$, $(x, y) \in X$. What is the quotient Riemann surface $\widehat{X}/\mu_N$? (hint : consider the meromorphic function x^N on $\widehat{X}$).

h) For the same Fermat curve, consider the diagonal action of the group $\mu_N \times \mu_N$. What is $\widehat{X}/(\mu_N \times \mu_N)$?

i) Let G be a discrete group, and $G \times X \to X$ a continuous action of G on a locally compact Hausdorff topological space. Show that if this action is proper, the stabilizers are finite and the quotient space X/G is Hausdorff.