Corrigé de l'exercice 2

Exercise 2.

a) *Show that the quotient Riemann surface $\mathbb{C}/\mathbb{Z}$ (with $\mathbb{Z}$ acting naturally by translations) is isomorphic to $\mathbb{C}^*$ (hint : show that the map $\mathbf{e} : \mathbb{C} \to \mathbb{C}^*$, $\mathbf{e}(z) = \exp(2i\pi z)$ defines a biholomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$).*

The map $\mathbf{e}$ is holomorphic and surjective, and the preimages of points (its "fibers") are the orbits of the action of $\mathbb{Z}$ on $\mathbb{C}$ $(k, z) \mapsto k + z$, which is proper. So $\mathbf{e} = f \circ p$ with $p : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ the quotient map, and $f : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$ a bijective holomorphic map of Riemann surfaces, hence a biholomorphism.

b) *Let $H = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$. Show that the quotient Riemann surface $H/\mathbb{Z}$ (with $\mathbb{Z}$ acting naturally by translations) is isomorphic to the punctured unit disc $\mathbb{D}^* = \{q \in \mathbb{C}^*; |q| < 1\}$.*

The restriction $\mathbf{e}|_H$ is holomorphic and surjects to $\mathbb{D}^*$, with fibers the orbits of the action of $\mathbb{Z}$ on H, which is proper (as in general the restriction of a proper action to an invariant subset). Hence as in the previous question, it defines a biholomorphism $H/\mathbb{Z} \to \mathbb{D}^*$.

c) *Let $\mu_n \subset \mathbb{C}^*$ be the group of n-th roots of 1, acting on $\mathbb{C}$ by multiplication. Show that the quotient Riemann surface $\mathbb{C}/\mu_n$ is isomorphic to $\mathbb{C}$.*

Since μ_n is finite, its (continuous) action is proper. Consider the map $p_n : \mathbb{C} \to \mathbb{C}$ defined by $p_n(z) = z^n$. It is holomorphic and surjective, and its fibers are the orbits of μ_n on $\mathbb{C}$. Hence it defines a biholomorphism $\mathbb{C}/\mu_n \to \mathbb{C}$.

d) *Show that the action of $\mathbb{Z}$ on $\mathrm{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ generated by $z \mapsto 2z$ is not proper. Show that it is proper on the complement of the fixed points. What is the corresponding quotient ?*

A proper action of a discrete group necessarily has finite stabilizers, but here 0 and ∞ have stabilizer $\mathbb{Z}$.

Restricting the action to $\mathbb{C}^* = \mathrm{P}^1(\mathbb{C}) \setminus \{0, \infty\}$, it becomes proper because two compact subsets of $\mathbb{C}^*$ are both contained in an annulus $A_R = \{z \in \mathbb{C} | 1/R \leq |z| \leq R\}$, and $2^k A_R \cap A_R \neq \emptyset$ implies $2^{\pm k} \leq R^2$.

The target of the quotient map $p : \mathbb{C}^* \to \mathbb{C}^*/2^{\mathbb{Z}}$ is compact, since A_R maps surjectively to it for $R \geq \sqrt{2}$.

To exhibit the quotient as an elliptic curve, consider the lattice $\Lambda = \log(2)\mathbb{Z} + 2i\pi\mathbb{Z} \subset \mathbb{C}$. Then $p \circ \exp : \mathbb{C} \to \mathbb{C}^*/2^{\mathbb{Z}}$ is surjective with fibers the orbits of the (proper) action of Λ on $\mathbb{C}$ (by addition), hence defines an isomorphism $\mathbb{C}/\Lambda \to \mathbb{C}^*/2^{\mathbb{Z}}$.

e) *Same as the previous question, but for $z \mapsto z + 1$ on $\mathrm{P}^1(\mathbb{C})$.*

Here the action of $\mathbb{Z}$ fixes ∞, hence is not proper. On the complement $\mathbb{C} = \mathrm{P}^1(\mathbb{C}) \setminus \{\infty\}$, we already know that the action is proper, with quotient isomorphic to $\mathbb{C}^*$.

f) *What about the action of μ_n (by multiplication) on $\mathrm{P}^1(\mathbb{C})$?*

Since μ_n is finite, the action is proper, and the map $z \mapsto z^n$ from $\mathrm{P}^1(\mathbb{C})$ to itself gives an isomorphism $\mathrm{P}^1(\mathbb{C})/\mu_n \to \mathrm{P}^1(\mathbb{C})$.

g) *Consider the action of μ_N on the compact Fermat curve $\widehat{X}$ of exercice 1, given on its affine part X by $\lambda \cdot (x, y) = (\lambda x, y)$, $\lambda^N = 1$, $(x, y) \in X$. What is the quotient Riemann surface $\widehat{X}/\mu_N$? (wrong hint : consider the meromorphic function x^N on $\widehat{X}$).*

The curve $\widehat{X}$ is constructed by gluing $X \subset \mathbb{C}^2$ to $X^* = \{(u, v) \in \mathbb{C}^2 \mid 1 + v^N = u^N\}$ (viewed as *disjoint*) with the biholomorphism $\varphi : (x, y) \mapsto (u, v) = (1/x, y/x)$ from $X \cap (\mathbb{C}^* \times \mathbb{C})$ to $X^* \cap (\mathbb{C}^* \times \mathbb{C})$. Formally, $\widehat{X}$ is the quotient of the disjoint union $X \sqcup X^*$ by the equivalence relation $(x, y) \sim \varphi(x, y)$.

The action $\lambda \cdot (x, y) = (\lambda x, y)$ of μ_N on X matches via φ with the action $\lambda \cdot (u, v) = (u/\lambda, v/lambda)$ on X^*. They are both holomorphic, as induced by complex linear actions of μ_N on $\mathbb{C}^2$.

Hence μ_N acts holomorphically on $\widehat{X}$, and Since μ_N is finite, this action is proper.
The *second* projection map $y : X \to \mathbb{C}$ extends to a holomorphic map $g : \widehat{X} \to \mathrm{P}^1(\mathbb{C})$ which on X^* is given by $(u, v) \mapsto v/u$. It has N simple poles in $\widehat{X}$, images of the points $(0, v)$ of $X^* \cap \{0\} \times \mathbb{C}$, solutions of $v^N = -1$.
The fiber $g^{-1}(z)$ over $z \in \mathbb{C}$ is the set of points $(x, z) \in X$, solutions of $x^N = 1 - z^N$. It is an orbit of the action of μ_N. And $g^{-1}(\infty)$ is also an orbit of this action, since $\lambda \cdot (0, v) = (0, v/\lambda)$.
Hence g factorizes as the quotient map $\widehat{X} \to \widehat{X}/\mu_N$ followed by an isomorphism $\widehat{X}/\mu_N \to \mathrm{P}^1(\mathbb{C})$.

h) *For the same Fermat curve, consider the "diagonal" (better : coordinatewise) action of the group* $\mu_N \times \mu_N$. *What is* $\widehat{X}/(\mu_N \times \mu_N)$ *?*

The group is still finite, hence any continuous action is proper.
We can *now* consider $x^N : X \to \mathbb{C}$, which extends to $h : \widehat{X} \to \mathrm{P}^1(\mathbb{C})$ of degree N^2, with fibers the orbits of $\mu_N \times \mu_N$ (this is easily verified on X and extends to the N points "at infinity"). Hence the quotient is isomorphic to $\mathrm{P}^1(\mathbb{C})$. Another proof would be to consider the action of the "second" μ_N on the quotient by the action of the first μ_N (studied in the previous question). This makes sense only because these two actions *commute*. And it exhibits $X/(\mu_N \times \mu_N)$ as the quotient of an action of μ_N on $\mathrm{P}^1(\mathbb{C})$, given by $(\lambda, y) \mapsto \lambda y$. Happily, we again find $\mathrm{P}^1(\mathbb{C})$.

i) *Let G be a discrete group, and $G \times X \to X$ a continuous action of G on a locally compact Hausdorff topological space. Show that if this action is proper, the stabilizers are finite and the quotient space X/G is Hausdorff.*

Properness of the action is (by definition) properness of the map $(g, x) \mapsto (g.x, x)$, the "graph map" $\varphi : G \times X \to X \times X$.
The stabilizers G_x $(x \in X)$ are finite, because $\varphi^{-1}((x, x)) = G_x \times \{x\}$ is compact and G discrete.
To show that X/G is Hausdorff is to show that for any two distinct (hence disjoint) orbits $G \cdot x$, $G \cdot y$ in X there are (open if we wish so) neighbourhoods U of x and V of y in X such that their "saturations" $G \cdot U$ and $G \cdot V$ are disjoint. Note that $G \cdot U = \cup_g g \cdot U$ is open if U is, and that it equals $p^{-1}(p(U))$ so that $p(U)$ is an open subset of X/G containing $p(x)$ (and similarly for $G \cdot V$).
The sets $G \cdot U$, $G \cdot V$ are not disjoint if and only if there is $g \in G$ with $g \cdot U \cap V \neq \emptyset$. But X being locally compact, we can choose compact neighbourhoods $U' \subset U$ of x, $V' \subset V$ of y. Then by properness there are only finitely many g with $g \cdot U'$ intersecting V', say $g_1, \ldots, g_n$. We have $g_i \cdot x \neq y$ for $i = 1, \ldots, n$ by assumption, and by Hausdorffness of X we obtain smaller (open) neighbourhoods $U_i'' \subset U'$ of x, $V_i'' \subset V'$ of y such that $g_i \cdot U_i'' \cap V_i'' = \emptyset$ for $i = 1, \ldots, n$. Then $U'' = \cap_i U_i''$, $V'' = \cap_i V_i''$ verify $G \cdot U'' \cap G \cdot V'' = \emptyset$.